# Exact Nonlinear Decomposition of Ideal-MHD Waves Using Eigenenergies

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# Why do we need to decompose MHD waves?

- MHD waves are ubiquitous in plasma systems such as stars or fusion reactors
- MHD waves have long been deemed important in heating of the solar chromosphere and corona, and even as an igniter for reconnection events
- The three fundamental MHD modes have different qualities with implications for wave-energy transport
- In any realistic setting, there are many physical processes that couple Alfvén, fast, and slow modes into mixed-mode MHD waves, therefore, teasing out the modes becomes complicated
- We often need to know which mode is dominating in a given wave-field to correctly analyze simulation or observational data

# Current decomposition methods

Most of the current methods are based on the asymptotic behaviors of MHD waves where they are less strongly coupled, i.e., in the extreme cases of $\beta = 0$ and $\beta \to \infty$ (e.g., Yadav et. al. 2022). They usually rely on

- Magnetic field line tracing
- Local asymptotic polarization vector of MHD waves

**Therefore, they are inexact and restricted to certain regions, while often more interesting stuff happens where these schemes break down.**

# Current decomposition methods

Some other studies use a combination of **asymptotic behaviors**, **dispersion diagrams**, and the **magnetic** and **acoustic energy** contents of mixed-mode waves for mode identification (see **Tarr et. al. 2017**, **Raboonik and Cally 2019 and 2021**).

Lastly, estimating the **speed of disturbances** and using time-distance plots also provide another rough means of mode identification, especially in analyzing observational data.

# MHD Equations

$$
\left\{
\begin{array}{l}
\partial_t \rho + \nabla.(\rho \boldsymbol{v}) = 0 \\
\partial_t \boldsymbol{v} + \boldsymbol{v}.\nabla \boldsymbol{v} + \dfrac{1}{\rho}(\nabla p + \boldsymbol{j} \times \boldsymbol{B}) = 0 \\
\partial_t \boldsymbol{B} - \boldsymbol{B}.\nabla \boldsymbol{v} - \boldsymbol{B}\nabla.\boldsymbol{v} - \boldsymbol{v}.\nabla \boldsymbol{B} = 0 \\
\partial_t p + \boldsymbol{v}.\nabla p + \gamma p \nabla.\boldsymbol{v} = 0
\end{array}
\right.
$$

$$
\boxed{\partial_t \boldsymbol{P} + M_q \partial_q \boldsymbol{P} = 0}
$$

# The eigensystem of ideal-MHD equations

$$\mathrm{M}_q = R_q \Lambda_q L_q$$

$$\Lambda_q = \mathrm{diag}(\underbrace{v_q}_{\text{Field divergence}}, \underbrace{v_q}_{\text{Entropy}}, \underbrace{v_q - a_q, v_q + a_q}_{\text{Alfvén}}, \underbrace{v_q - c_{\mathrm{s},q}, v_q + c_{\mathrm{s},q}}_{\text{Slow}}, \underbrace{v_q - c_{\mathrm{f},q}, v_q + c_{\mathrm{f},q}}_{\text{Fast}})$$

$$a_q = \frac{B_q}{\sqrt{\mu_0 \rho}} \; ; \quad c_{\mathrm{f/s},q} = \frac{\sqrt{a^2 + c^2 \pm \sqrt{a^4 + c^4 + c^2\left(a^2 - 2a_q^2\right)}}}{\sqrt{2}}$$

# The Eigenenergy Decomposition Method (EEDM)

$$\mathrm{E}_{\mathrm{tot}} = \frac{1}{2}\rho v^2 + \frac{p}{\gamma - 1} + \frac{1}{\mu_0}\boldsymbol{B}\cdot\boldsymbol{B}$$

$$\partial_t E_{\mathrm{tot}} = \frac{1}{2}\rho_t v^2 + \rho \boldsymbol{v}_t\cdot\boldsymbol{v} + \frac{p_t}{\gamma - 1} + \frac{1}{\mu_0}\boldsymbol{B}_t\cdot\boldsymbol{B}$$

$$\partial_t \boldsymbol{P} + R_q \Lambda_q L_q \partial_q \boldsymbol{P} = 0$$

$$\boxed{\partial_t E_{\mathrm{tot}} = \sum_{q\in(x,y,z)} \left(\partial_t E_{\mathrm{div},q} + \partial_t E_{\mathrm{ent},q} + \partial_t E^-_{\mathrm{A},q} + \partial_t E^+_{\mathrm{A},q} + \partial_t E^-_{\mathrm{s},q} + \partial_t E^+_{\mathrm{s},q} + \partial_t E^-_{\mathrm{f},q} + \partial_t E^+_{\mathrm{f},q}\right)}$$

# Mode-decomposed wave energy

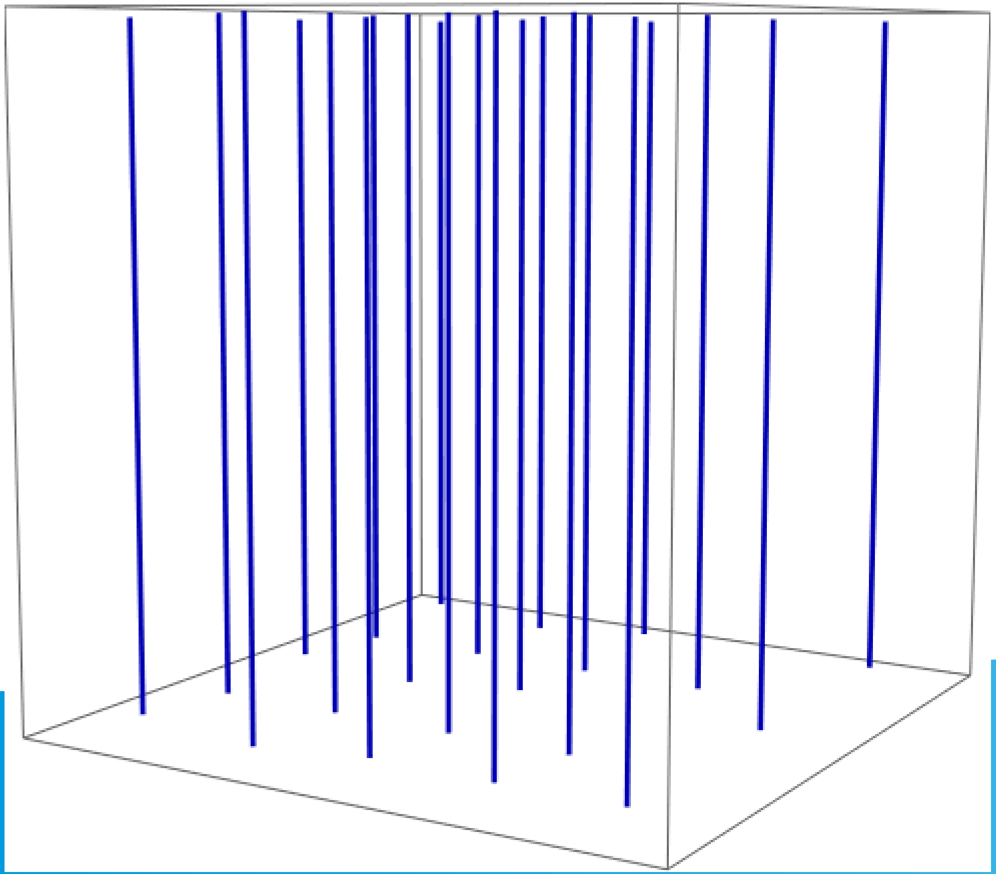
$$\boxed{\mathrm{E}_{\mathrm{tot}} = \underbrace{\sum_{q\in(x,y,z)} \left(E_{\mathrm{div},q} + E_{\mathrm{ent},q} + E^{-}_{\mathrm{A},q} + E^{+}_{\mathrm{A},q} + E^{-}_{\mathrm{s},q} + E^{+}_{\mathrm{s},q} + E^{-}_{\mathrm{f},q} + E^{+}_{\mathrm{f},q}\right)}^{E_{\mathrm{rec}}} + f(x,y,z)}$$

$$\mathrm{E}_{\mathrm{tot}} = \frac{1}{2}\rho v^2 + \frac{p}{\gamma-1} + \frac{1}{\mu_0}\boldsymbol{B}\cdot\boldsymbol{B}$$

$$= E_{\mathrm{orig}} + E_0$$

# Simulation Ia:
# High-beta case ($\beta = 9$)

Background field:



Driver: torsional Alfven wave

$$\boldsymbol{B}_\perp(z,t) = B_\perp(z,t)\left(\cos\left(kz-\omega t+\varphi\right), \sin\left(kz-\omega t+\varphi\right)\right),$$

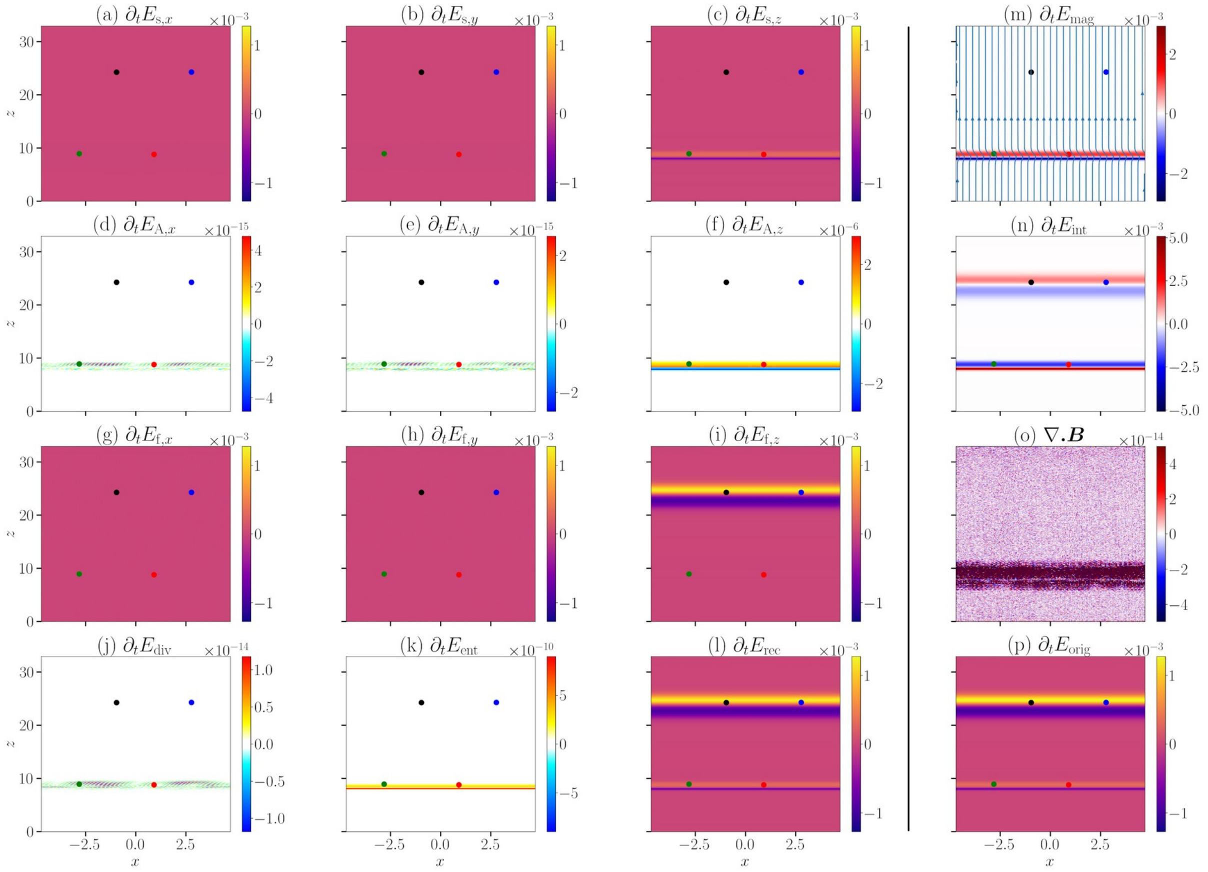
$$\boldsymbol{v}_\perp(z,t) = -\frac{\boldsymbol{B}_\perp(z,t)}{B_0}a_0,$$

$$\omega = a_0 k,$$

$$B_\perp(z,t) = A\exp\left(-\frac{(z-a_0t-z_0)^2}{2\sigma^2}\right)$$

Turkmani & Torkelsson (2004)

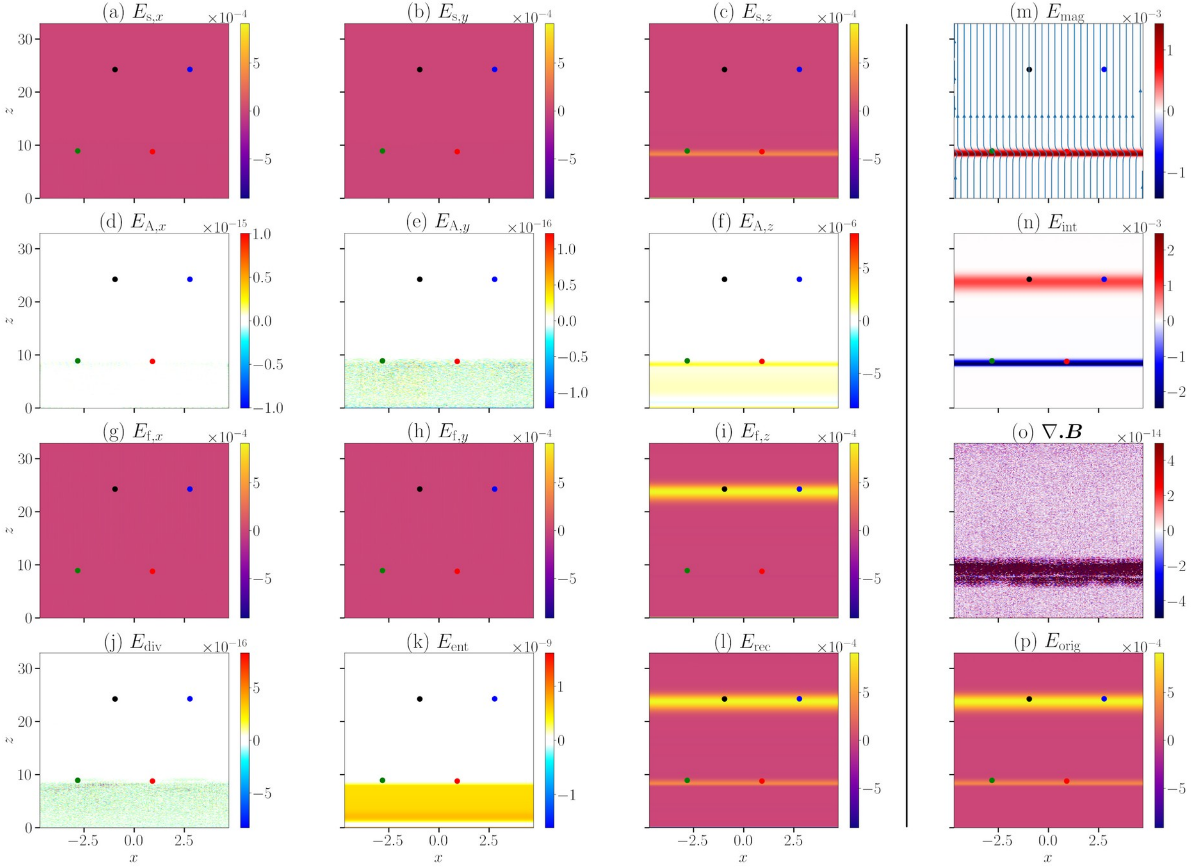
$t = 17.95, y =$ -0.024






https://arxiv.org/abs/2402.05327

$t$ = 17.95, $y$ = -0.024

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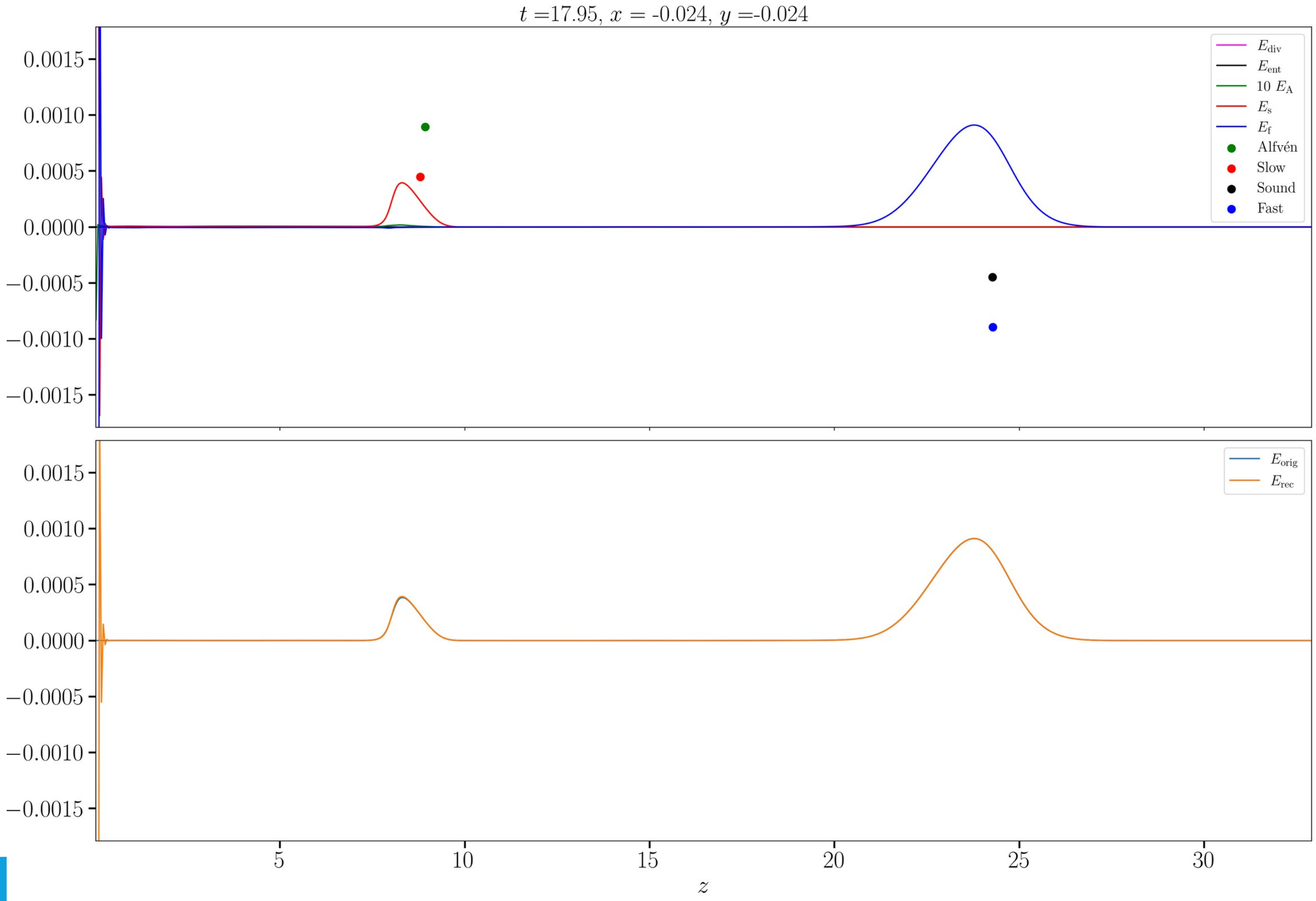


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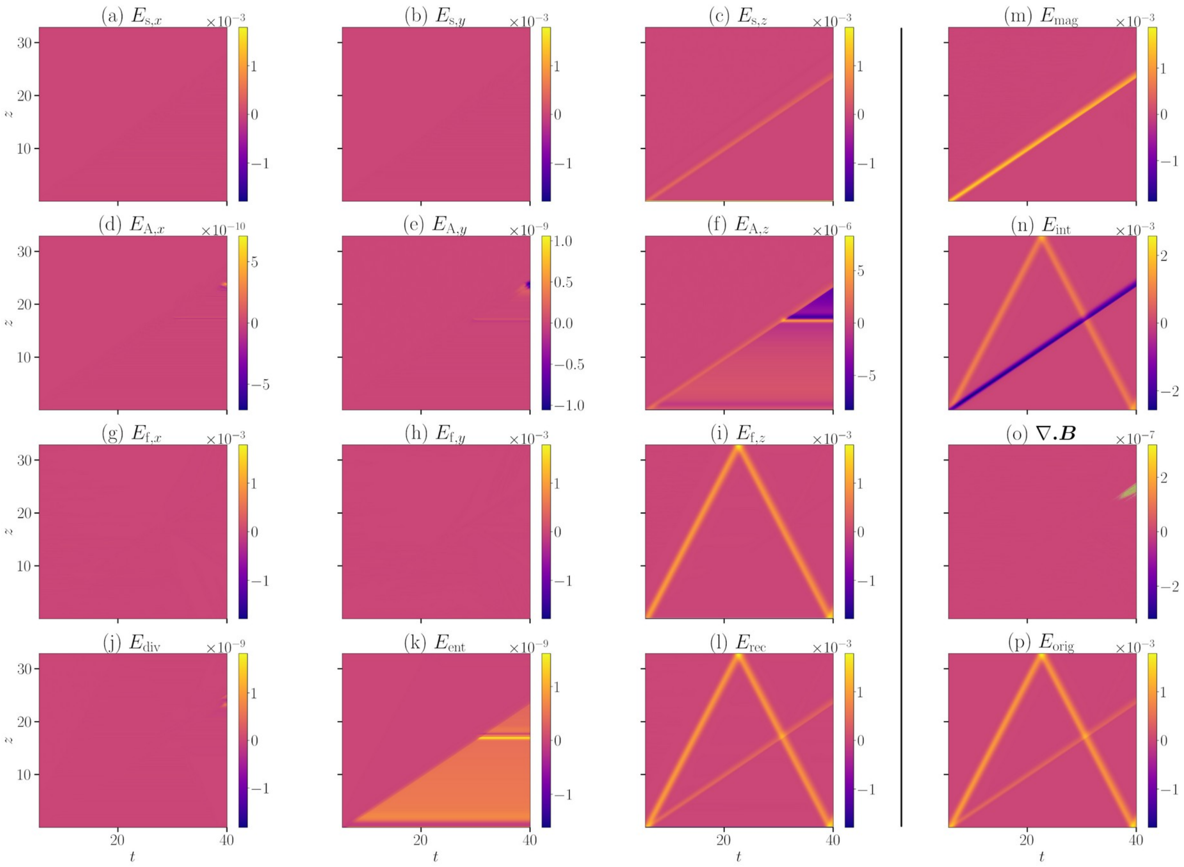
$x = \text{-}0.024, y = \text{-}0.024$

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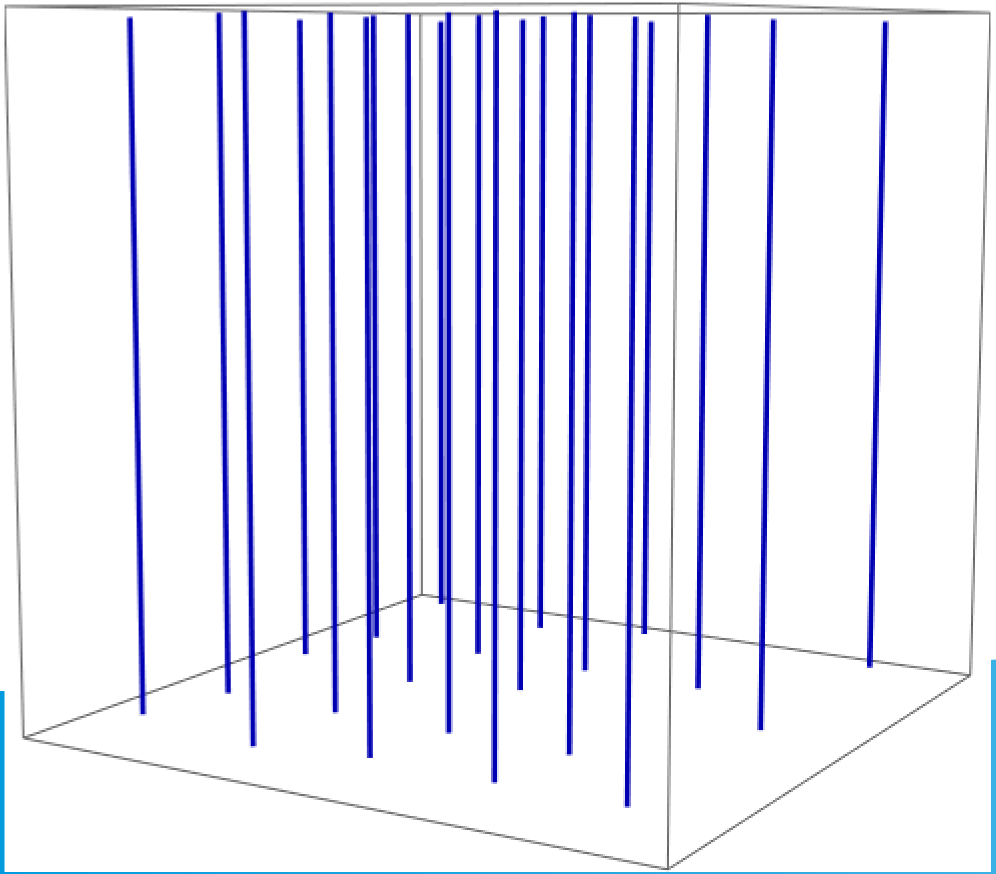
# Simulation Ib:
# Low-beta case ($\beta = 0.54$)

Background field



Driver: torsional Alfven wave

$$\boldsymbol{B}_\perp(z,t) = B_\perp(z,t)\left(\cos\left(kz - \omega t + \varphi\right), \sin\left(kz - \omega t + \varphi\right)\right),$$

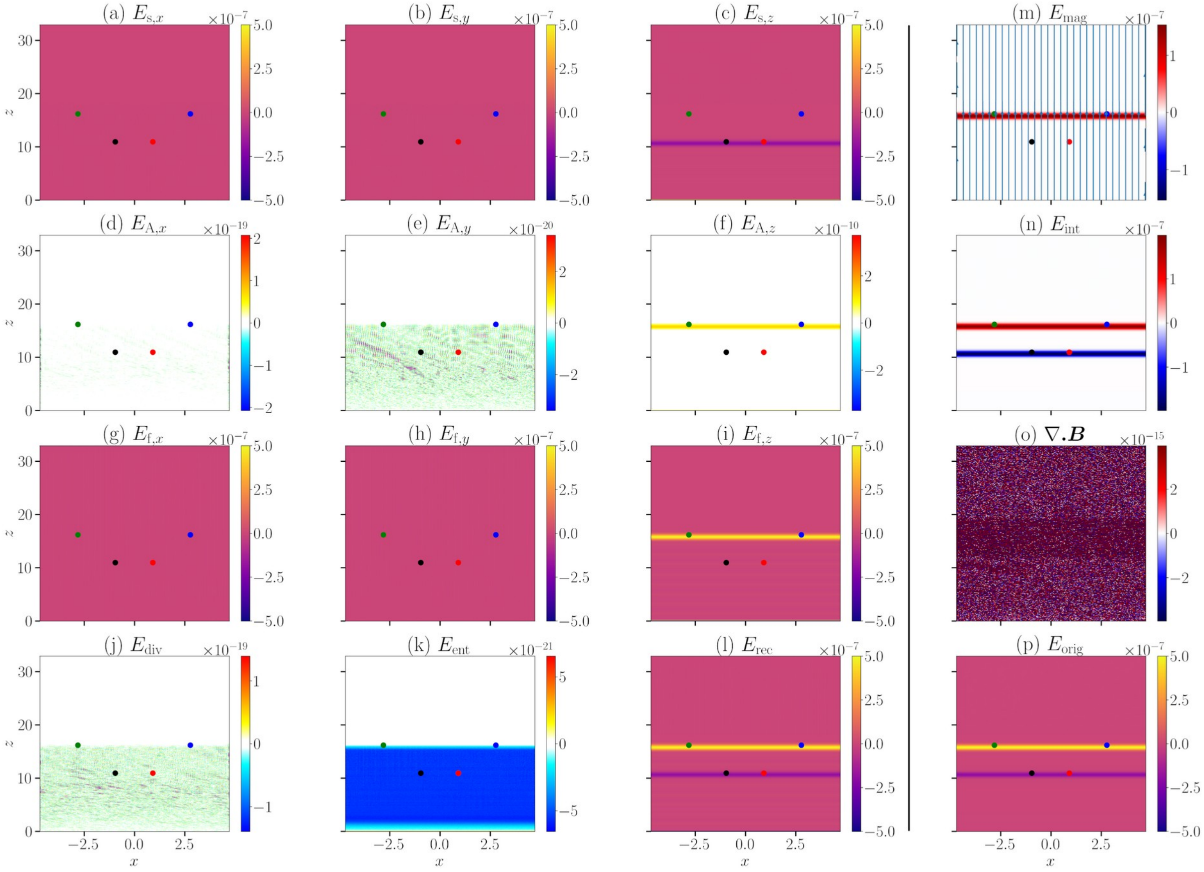
$$\boldsymbol{v}_\perp(z,t) = -\frac{\boldsymbol{B}_\perp(z,t)}{B_0}a_0,$$

$$\omega = a_0 k,$$

$$B_\perp(z,t) = A\exp\left(-\frac{(z - a_0 t - z_0)^2}{2\sigma^2}\right)$$

Turkmani & Torkelsson (2004)

$t = 0.99$, $y =$ -0.024

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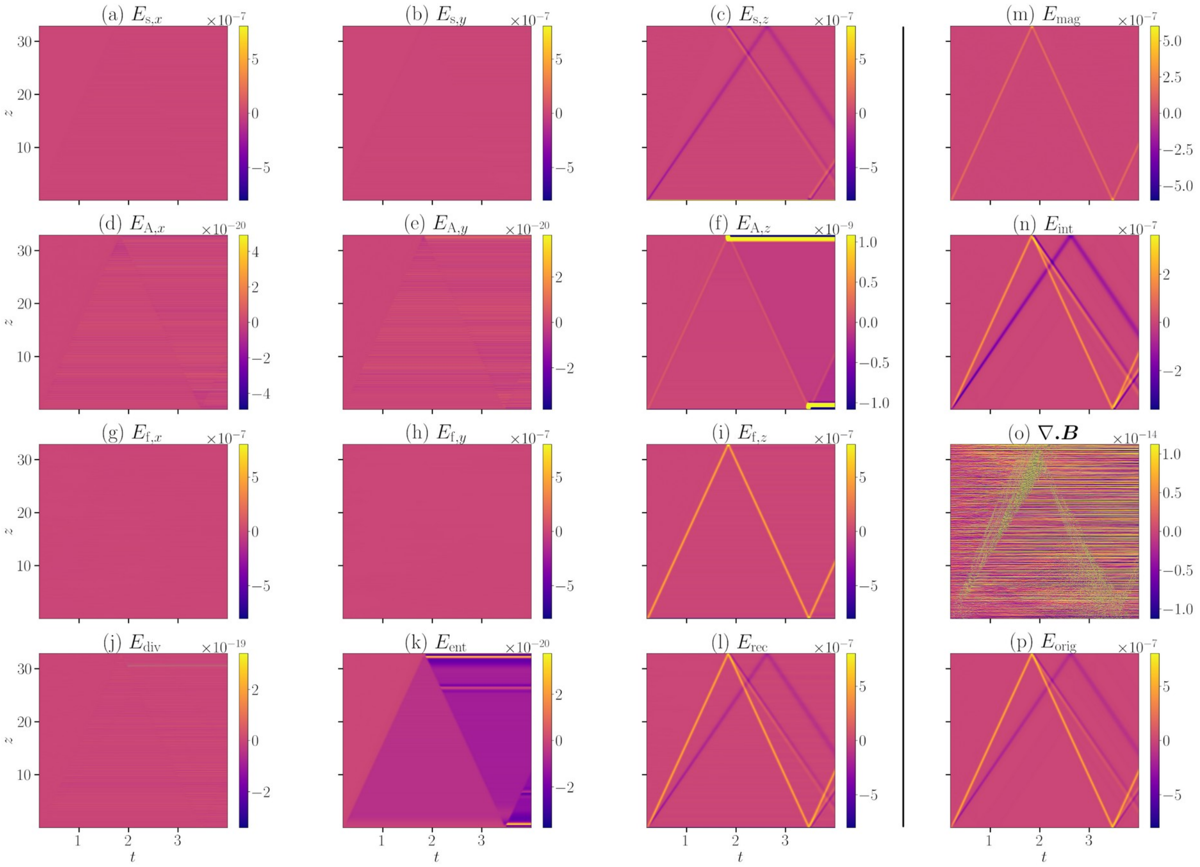
$x = \text{-}0.024, y = \text{-}0.024$

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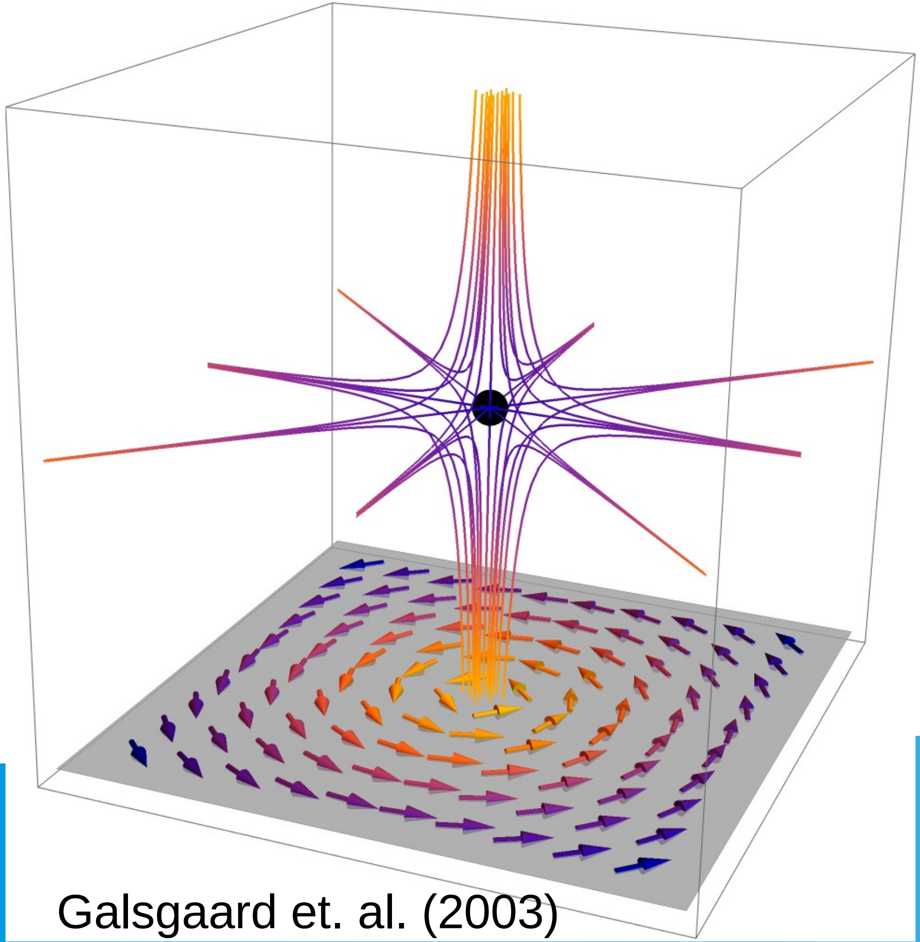
# Simulation II:
# 3D wave-null interaction



Galsgaard et. al. (2003)

Driver:

$$\boldsymbol{v} = v_0 \exp\left(-\frac{x^2+y^2}{2\sigma_R^2}\right) \exp\left(-\frac{t-t_0}{2\sigma_P^2}\right)(y,-x,0)\,,$$

$$\boldsymbol{B} = \boldsymbol{B}_0 - \sqrt{\rho_0}\boldsymbol{v}$$

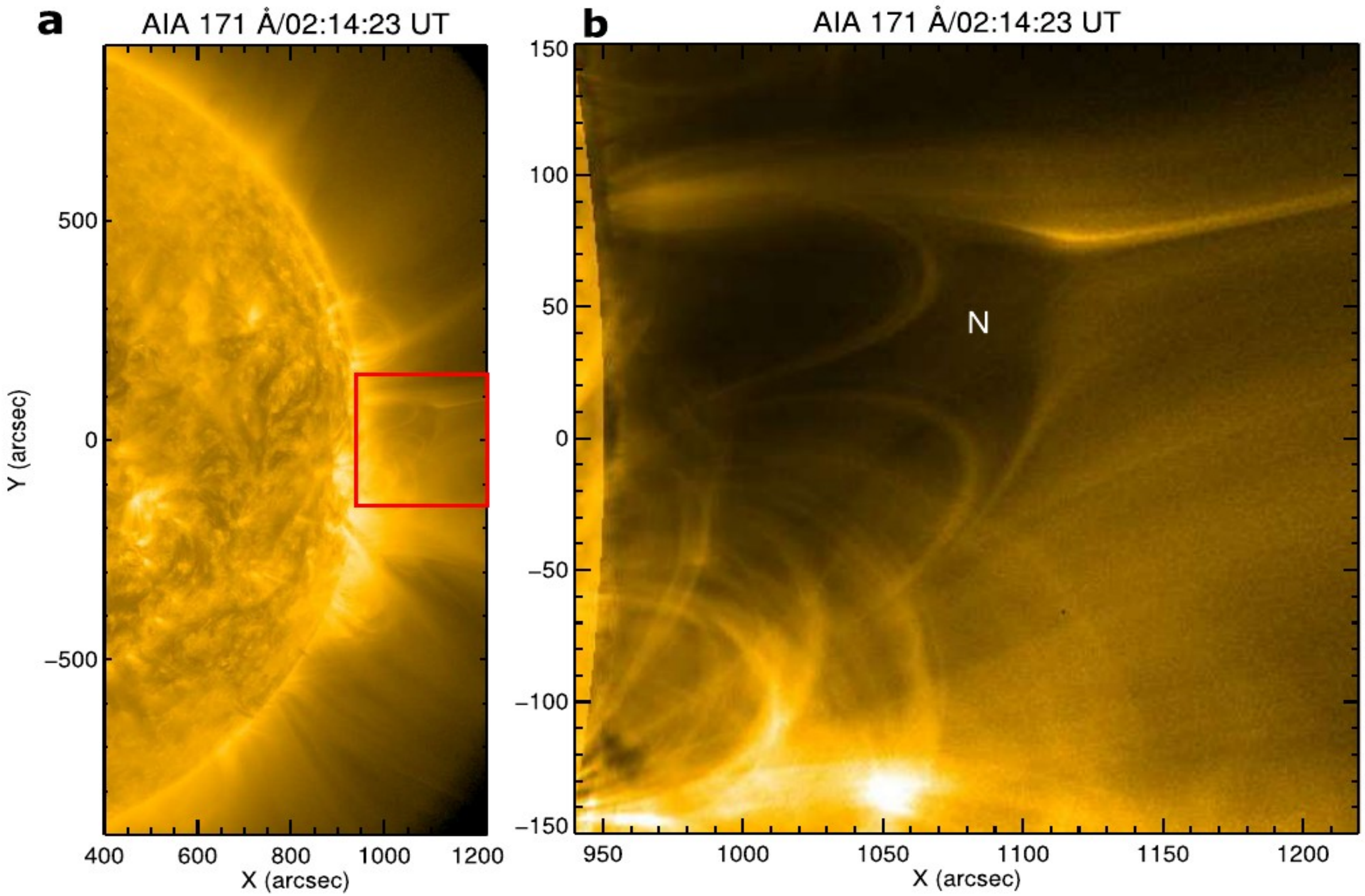
# A recent observational study

ABSTRACT

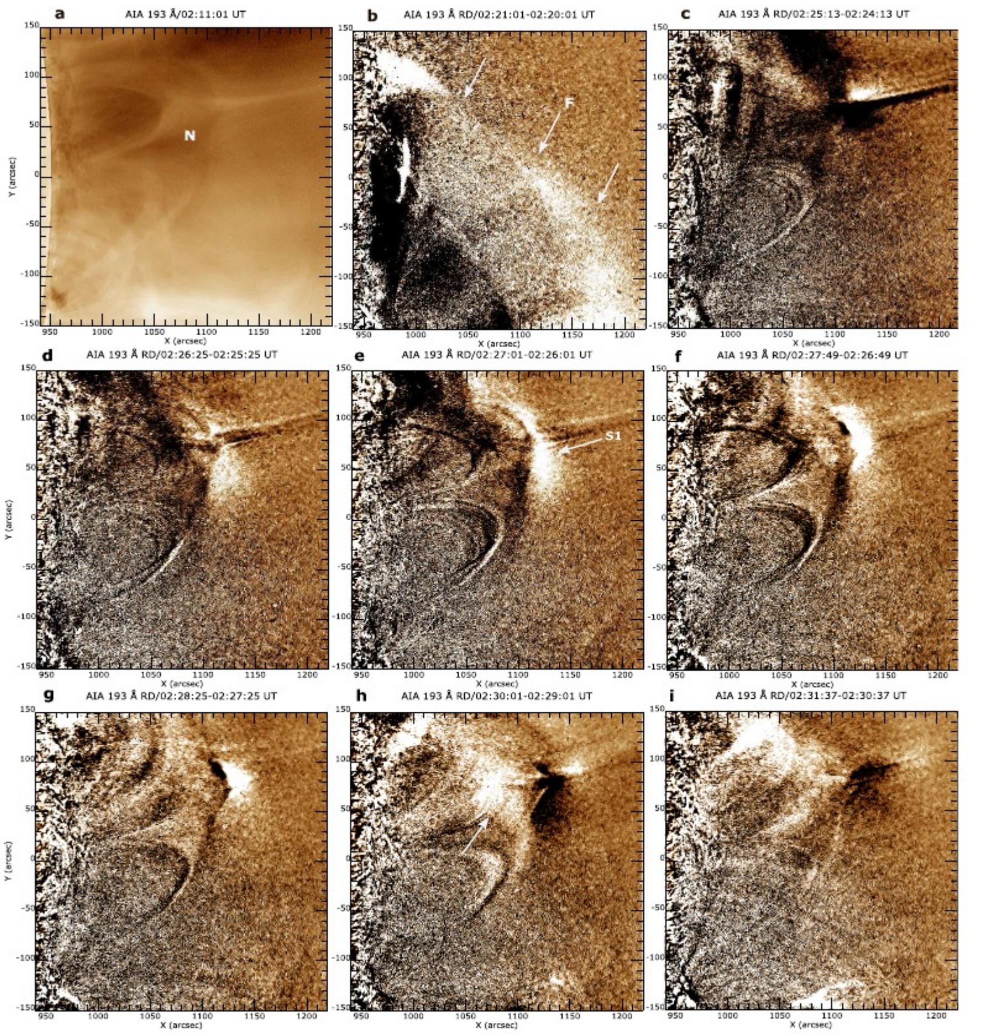
Mutual conversion of various kinds of magnetohydrodynamic (MHD) waves can have profound impacts on wave propagation, energy transfer, and heating of the solar chromosphere and corona. Mode conversion occurs when an MHD wave travels through a region where the Alfvén and sound speeds are equal (e.g., a 3D magnetic null point). Here we report the first EUV imaging of mode conversion from a fast-mode to a slow-mode MHD wave near a 3D null point using Solar Dynamics Observatory/Atmospheric Imaging Assembly (SDO/AIA) observations. An incident fast EUV wavefront associated with an adjacent eruptive flare propagates laterally through a neighboring pseudostreamer. Shortly after the passage of the fast EUV wave through the null point, a slow-mode wave appears near the null that propagates upward along the open structures and simultaneously downward along the separatrix encompassing the fan loops of the pseudostreamer base. These observations suggest the existence of mode conversion near 3D nulls in the solar corona, as predicted by theory and MHD simulations. Moreover, we observe decaying transverse oscillations in both the open and closed structures of the pseudostreamer, along with quasiperiodic type III radio bursts indicative of repetitive episodes of electron acceleration.

Kumar et. al. (2024)

https://arxiv.org/abs/2403.02250

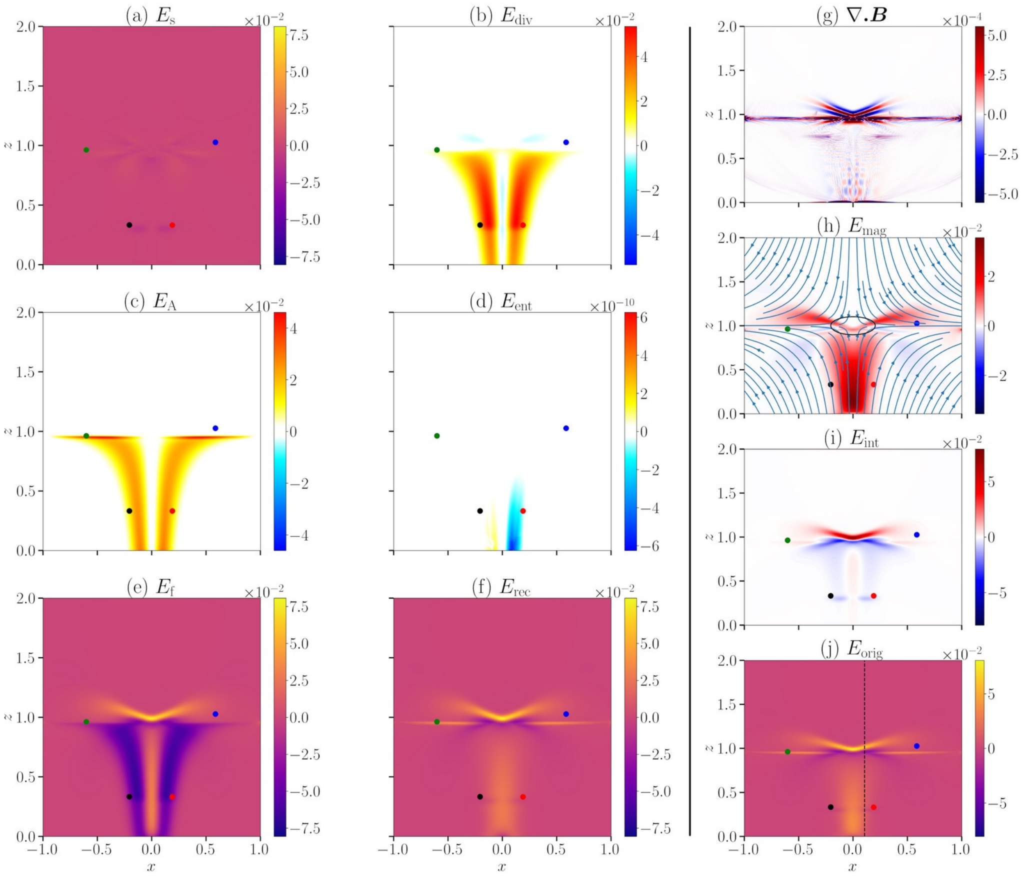



https://arxiv.org/abs/2403.02250




https://arxiv.org/abs/2403.02250

$t = 1.34,\ y = 0.003$

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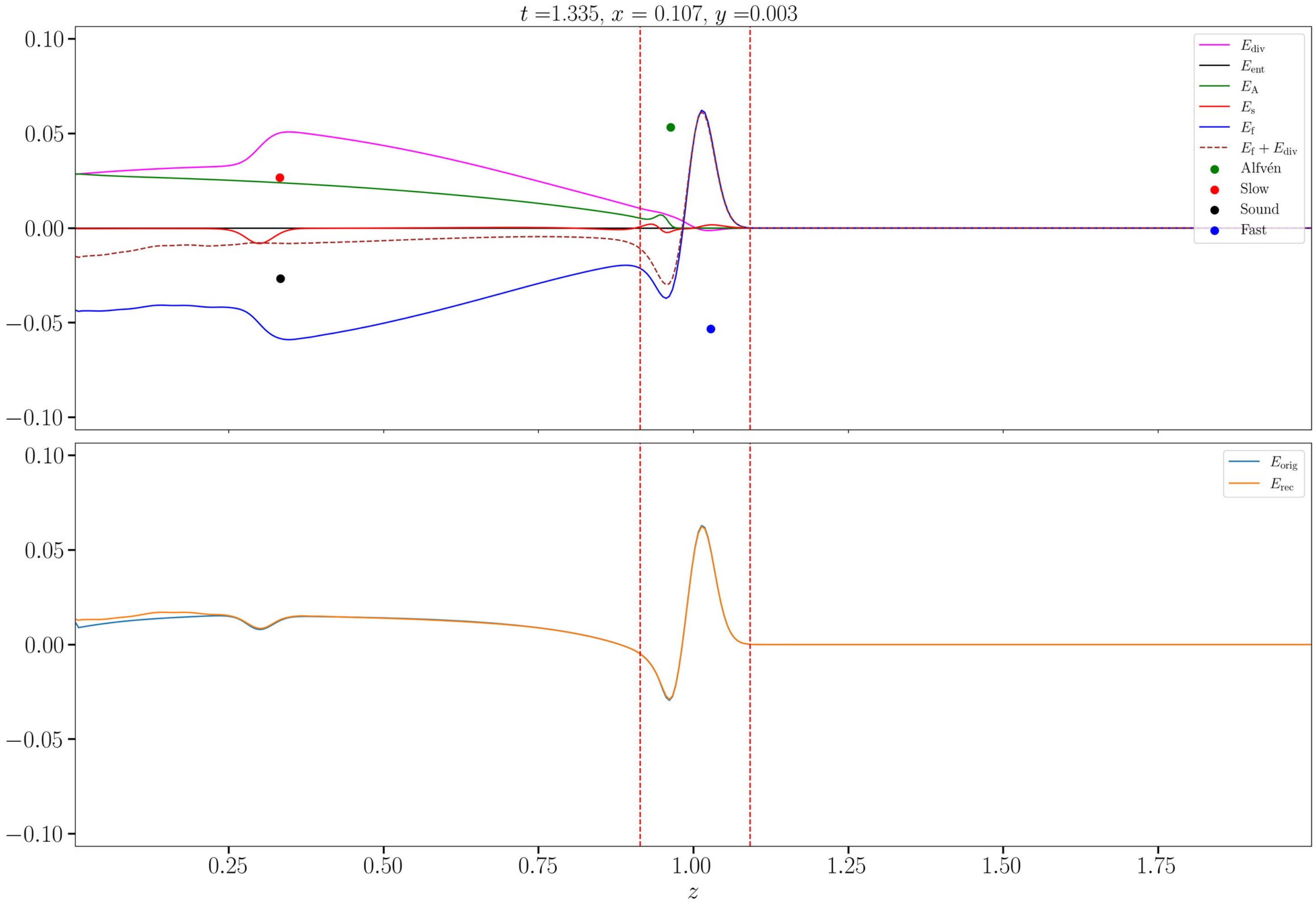


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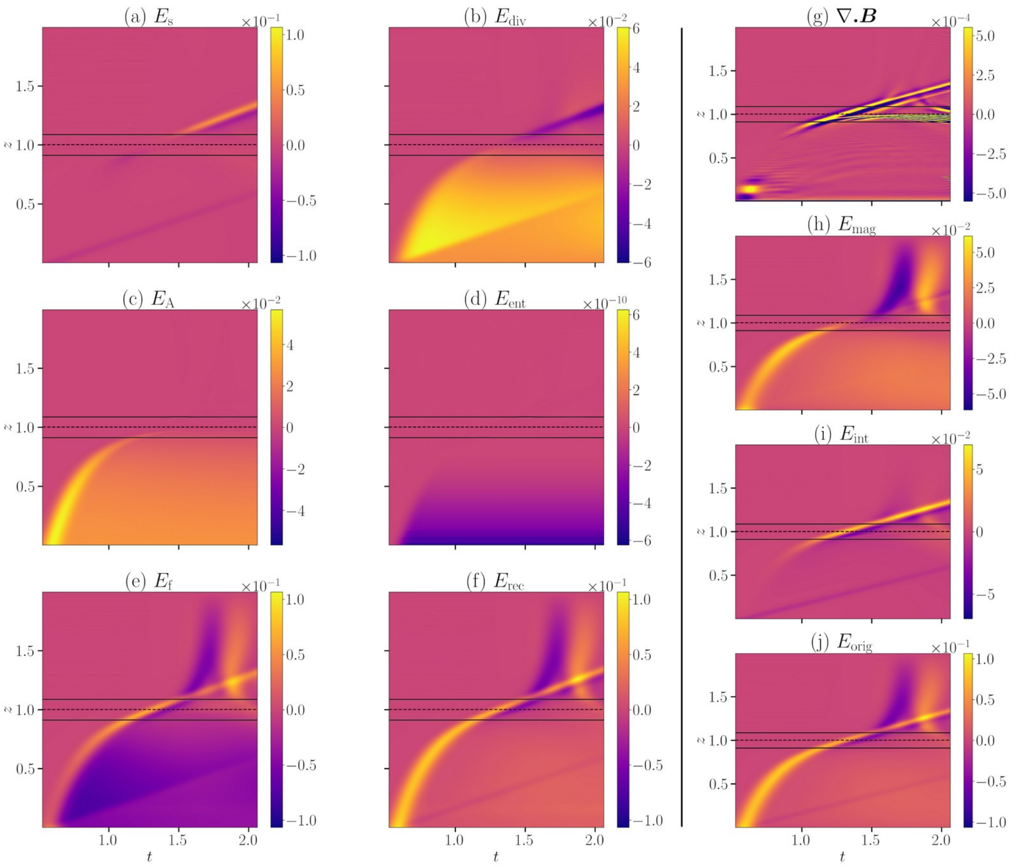
$x = 0.107,\ y = 0.003$




https://arxiv.org/abs/2402.05327

# Conclusion

- MHD waves are abundant in plasmas and play a vital role in determining the global dynamics of the system. Therefore, understanding them can provide insights into the transport of energy as well as other plasma phenomena.

- The Eigenenergy Decomposition Method (EEDM) provides a rigorous mathematical tool for exact decomposition of complex and fully nonlinear ideal-MHD waves in terms of their energy content.

- EEDM can be applied to any data of sufficient cadence as long as the state-vector is fully known in the region of interest.

- EEDM remains approximately valid where there is other physics beyond ideal-MHD (e.g. dissipative effects such as magnetic diffusion etc), as long as the effects are sufficiently small.