A Fibonacci-like universe expansion on time-scale

Octavian Postavaru

CiTi, DMM - National University of Science and Technology Politehnica Bucharest

September 27, 2023

Motivation

Time-scale (Hilger)

- generalize continuous analysis and discrete analysis
- unifies differential and difference equations

Fibonacci

- describes patterns: leaf, tree, galaxies, shell of the snail, ...
- using the light curves of the Kepler telescope, it can be pointed out that the ratio between the primary and secondary pulsation of some stars satisfies the golden mean

O. Postavaru and A. Toma, *A Fibonacci-like universe expansion on time-scale*, Chaos, Solitons and Fractals **154** (2022) 111619.

Planck scale

Fundamental question: is nature discrete or continous?

Table 1: Modern values for Planck's original choice of quantities

Name	Dimension	Expression	Value (SI units)
Planck length	length (L)	$l_\mathrm{P} = \sqrt{\frac{\hbar G}{c^3}}$	$1.616\,255(18) \times 10^{-35}$ m[7]
Planck mass	mass (M)	$m_\mathrm{P} = \sqrt{\frac{\hbar c}{G}}$	$2.176\,434(24) \times 10^{-8}$ kg[8]
Planck time	time (T)	$t_\mathrm{P} = \sqrt{\frac{\hbar G}{c^5}}$	$5.391\,247(60) \times 10^{-44}$ s[9]
Planck temperature	temperature (Θ)	$T_\mathrm{P} = \sqrt{\frac{\hbar c^5}{G k_\mathrm{B}^2}}$	$1.416\,784(16) \times 10^{32}$ K[10]

$$\Delta x \Delta p \geq \hbar/2$$

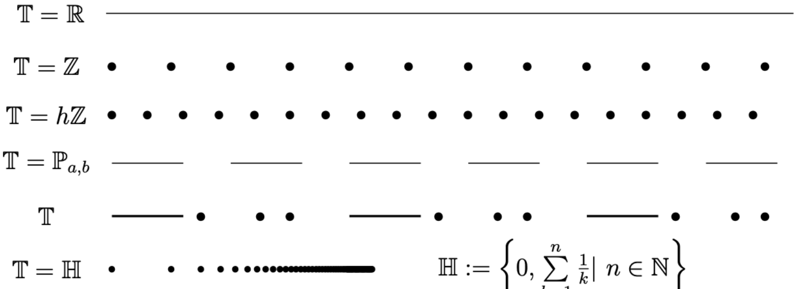
Yang-Mills problem

MILLENNIUM PRIZE SERIES: ***The Millennium Prize Problems are seven mathematics problems laid out by the Clay Mathematics Institute in 2000. They're not easy—_*** **a correct solution to any one results in a US$1,000,000 prize being awarded by the institute_.**

- mass gap: lattice
- confinement: lattice
- chiral symmetry breaking: continuous theory

Time-scale

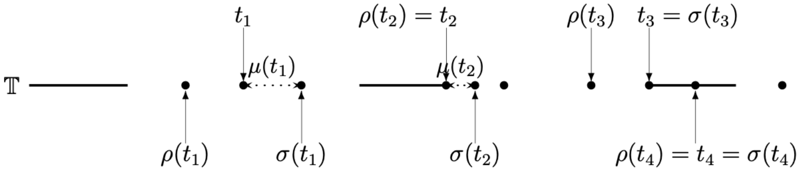
Examples:



$$\mathbb{H} := \left\{ 0, \sum_{k=1}^{n} \frac{1}{k} \middle| \; n \in \mathbb{N} \right\}$$

Time-scale calculus

the *forward jump* operator: $\sigma : \mathbb{T} \to \mathbb{T}$, $\sigma(t) := \inf\{s \in \mathbb{T} | s > t\}$
the *graininess* function: $\mu : \mathbb{T} \to [0, \infty)$, $\mu(t) := \sigma(t) - t$



$$f^{\Delta}(\tau) = \frac{\Delta f}{\Delta \tau} = \lim_{s \to \tau} \frac{f(\sigma(\tau)) - f(s)}{\sigma(\tau) - s}$$

Examples:
$\mathbb{T} = \mathbb{R}$, we have $f^{\Delta}(\tau) = f'(\tau)$
$\mathbb{T} = \mathbb{Z}$, then $f^{\Delta}(\tau) = f(\tau + 1) - f(\tau)$

Fibonacci

The Fibonacci sequence is given by the formula

$$F_t = F_{t-1} + F_{t-2} \,,$$

$F_0 = 0$, $F_1 = 1$

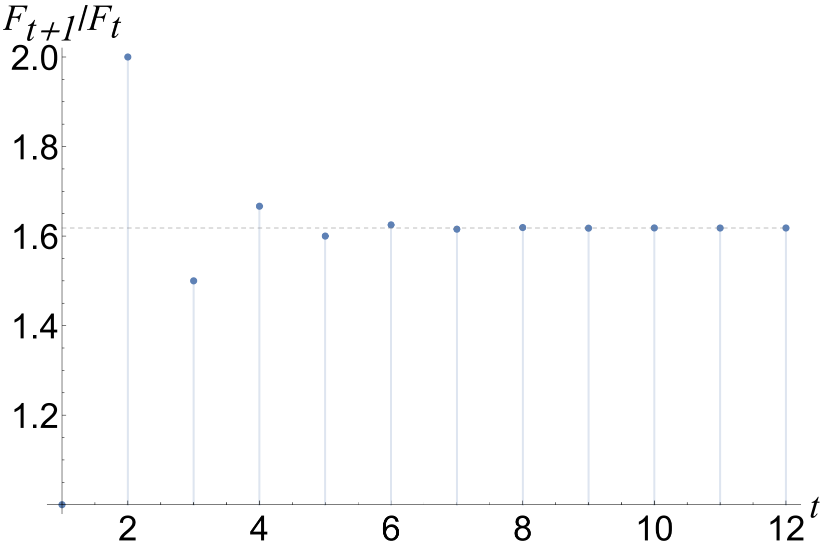
$$\frac{F_{t+1}}{F_t} \approx \varphi = \frac{1+\sqrt{5}}{2} \approx 1.61803398\ldots \,,$$

$t \approx 7$ the approximation becomes equality
Binet's formula

$$F(t) = \frac{\varphi^t - (-\varphi)^{-t}}{\sqrt{5}} \,.$$

throughout the work we use $t \in \mathbb{T}$

Fibonacci



Lagrangian formalism on time-scale

We propose the following Lagrangian

$$\mathcal{L}\left(F(t)^{\sigma}, F(t)^{\Delta}\right) = \frac{1}{2}\left(F(t)^{\Delta}\right)^{2} + \frac{C}{2}\left(F(t)^{\sigma}\right)^{2} ,$$

where $F(t)^{\sigma} \equiv F(\sigma(t))$.
The Euler-Lagrange equation* is

$$\frac{\Delta}{\Delta t}\left(\frac{\partial \mathcal{L}}{\partial F^{\Delta}}\right) - \frac{\partial \mathcal{L}}{\partial F^{\sigma}} = 0 ,$$

and we get the equation of motion (analogy with inverted harmonic oscillator equation)

$$F(t)^{\Delta\Delta} - C\, F(t)^{\sigma} = 0 .$$

*O. Postavaru and A. Toma, *Symmetries for Nonconservative Field Theories on Time Scale*, Symmetry **13**(4) (2021) 552

FLRW on time-scale

The Friedmann equation

$$H^2 \equiv \left(\frac{a(t)^{\Delta}}{a(t)}\right)^2 = \frac{8\pi G}{3}\rho(t) - \frac{K}{a^2(t)} + \frac{\Lambda}{3}\,,$$

Raychaudhuri equation

$$\frac{a(t)^{\Delta\Delta}}{a(t)} = -\frac{4\pi G}{3}\left(\rho(t) + 3P(t)\right) + \frac{\Lambda}{3}\,,$$

and covariant conservation equation

$$(\rho(t))^{\Delta} + 3\,H\,\left(P(t) + \rho(t)\right) = 0\,.$$

Illustrative examples: discrete theory ($\mathbb{T} = \mathbb{Z}$)

$F(t)^{\Delta} = \Delta F(t) = F(t+1) - F(t)$, $t \in \mathbb{Z}$
Proposition: If $t \in \mathbb{Z}$, then

$$F(t)^{\Delta^k} = F(t-k)\,, \quad \text{with} \quad \Delta^k = \underbrace{\Delta\Delta\ldots\Delta}_{k \text{ times}}\,.$$

Proposition: If $t \in \mathbb{Z}$, then we have

$$F(t)^{\Delta\Delta} - (2-\varphi)F(t) = 0\,.$$

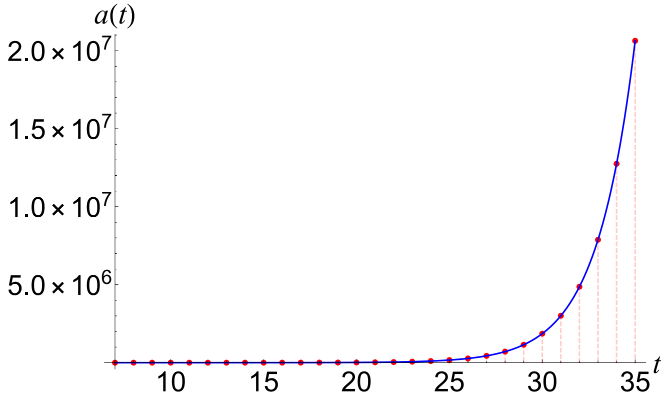
Proof:

$$\frac{F(t)^{\Delta\Delta}}{F(t)} = \frac{F(t-2)}{F(t-1)+F(t-2)} = 1 - \frac{F(t-1)}{F(t)} = 1 - \frac{1}{\varphi} = 2 - \varphi\,.$$

Discrete vs continous theory

Proposition: If $t \in \mathbb{T}$ and $\sqrt{\frac{\Lambda}{3}}\mu \ll 1$, with $\mu(t) = \sigma(t) - t$, then

$$a(t) = a(0)\,\exp\left(\sqrt{\frac{\Lambda}{3}}\,t\right).$$



Sketch of proof (FLRW)

Einstein's equation

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R - \Lambda g_{\mu\nu} = 8\pi G T_{\mu\nu} \,,$$

$R_{\mu\nu}$ is the Ricci tensor, R is the Ricci scalar, G is the universal gravitational constant, Λ cosmological constant, and $T_{\mu\nu}$ is the energy-momentum tensor.

For a perfect fluid, we can write

$$T_{00} = \rho(t) g_{00} \,, \quad T_{ii} = -p(t) g_{ii} \,.$$

The FLRW metric is written

$$ds^2 = c^2 dt^2 - a(t)^2 \left(\frac{dr^2}{1 - kr^2} + r^2 \left(d\theta^2 + \sin^2\theta d\phi^2 \right) \right) \,,$$

where (r, θ, ϕ) are spherical coordinates, k is the Gaussian curvature of the space, and $a(t)$ is known as scale factor.

Sketch of proof

Christoffel symbols

$$\Gamma^{\rho}_{\mu\nu} = \frac{1}{2} g^{\rho\sigma} \left(\partial_\mu g_{\nu\sigma} + \partial_\nu g_{\mu\sigma} - \partial_\sigma g_{\mu\nu} \right) ,$$

where $\rho\,,\sigma\,,\mu\,,\nu = \overline{0,3}$.

$$\begin{aligned}
\Gamma^0_{11} &= -\frac{1}{2} g^{00} (g_{11})^\Delta = \frac{1}{2} g^{00} \frac{(a^2)^\Delta}{1-kr^2} \approx \frac{aa^\Delta}{1-kr^2} \,, \\
\Gamma^0_{22} &= -\frac{1}{2} g^{00} (g_{22})^\Delta = \frac{1}{2} g^{00} (a^2)^\Delta r^2 \approx aa^\Delta r^2 \,, \\
\Gamma^0_{33} &= -\frac{1}{2} g^{00} (g_{33})^\Delta = \frac{1}{2} g^{00} (a^2)^\Delta r^2 \sin^2\theta \approx aa^\Delta r^2 \sin^2\theta \,, \\
\Gamma^1_{01} &= \Gamma^1_{10} = \frac{1}{2} g^{11} (g_{11})^\Delta = \frac{(a^2)^\Delta}{2a^2} \approx \frac{a^\Delta}{a} \,, \\
\Gamma^2_{02} &= \Gamma^2_{20} = \frac{1}{2} g^{22} (g_{22})^\Delta = \frac{(a^2)^\Delta}{2a^2} \approx \frac{a^\Delta}{a} \,,
\end{aligned}$$

(1)

Sketch of proof

$$\Gamma^3_{03} = \Gamma^3_{30} = \frac{1}{2} g^{33} (g_{33})^{\Delta} = \frac{(a^2)^{\Delta}}{2a^2} \approx \frac{a^{\Delta}}{a} \,,$$

$$\Gamma^1_{33} = -\frac{1}{2} g^{11} \partial_1 g_{33} = -r(1 - kr^2) \sin^2\theta \,,$$

$$\Gamma^3_{13} = \Gamma^3_{31} = \frac{1}{2} g^{33} \partial_1 g_{33} = \frac{1}{r} \,,$$

$$\Gamma^2_{33} = -\frac{1}{2} g^{22} \partial_2 g_{33} = -\sin\theta\cos\theta \,,$$

$$\Gamma^3_{23} = \Gamma^3_{32} = \frac{1}{2} g^{33} \partial_2 g_{33} = \frac{1}{\tan\theta} \,,$$

$$\Gamma^1_{22} = -\frac{1}{2} g^{11} \partial_1 g_{22} = -r(1 - kr^2) \,,$$

$$\Gamma^1_{11} = \frac{1}{2} g^{11} \partial_1 g_{11} = \frac{kr}{1 - kr^2} \,,$$

$$\Gamma^2_{12} = \Gamma^2_{21} = \frac{1}{2} g^{22} \partial_1 g_{22} = \frac{1}{r} \,,$$

Sketch of proof

Riemann tensor

$$R^{\rho}_{\sigma\mu\nu} = \partial_\mu \Gamma^{\rho}_{\nu\sigma} - \partial_\nu \Gamma^{\rho}_{\mu\sigma} + \Gamma^{\rho}_{\mu\lambda}\Gamma^{\lambda}_{\nu\sigma} - \Gamma^{\rho}_{\nu\lambda}\Gamma^{\lambda}_{\mu\sigma}\,,$$

where $\rho\,,\sigma\,,\mu\,,\nu\,,\lambda = \overline{0,3}$.

$$R_{00} = R^{\rho}_{0\rho 0}$$

$$\begin{aligned} R^{1}_{010} &= \partial_1 \Gamma^{1}_{00} - \left(\Gamma^{1}_{10}\right)^{\Delta} + \Gamma^{1}_{1\lambda}\Gamma^{\lambda}_{00} - \Gamma^{1}_{0\lambda}\Gamma^{\lambda}_{10} \\ &= -\left(\Gamma^{1}_{10}\right)^{\Delta} - \left(\Gamma^{1}_{00}\Gamma^{0}_{10} + \Gamma^{1}_{01}\Gamma^{1}_{10} + \Gamma^{1}_{02}\Gamma^{2}_{10} + \Gamma^{1}_{03}\Gamma^{3}_{10}\right) \\ &= -\left(\Gamma^{1}_{10}\right)^{\Delta} - \left(\Gamma^{1}_{10}\right)^{2} \approx -\frac{a^{\Delta\Delta}}{a}\,, \end{aligned}$$

Sketch of proof

The only components of the Ricci tensor that are different from 0 are

$$R_{00} = -3\frac{a^{\Delta\Delta}}{a}, \quad R_{ii} = -\frac{g_{ii}}{a^2}\left(aa^{\Delta\Delta} + 2(a^{\Delta})^2 + 2k\right),$$

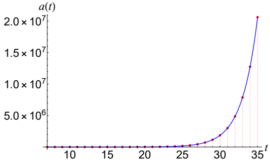
with $i = \overline{1,3}$.

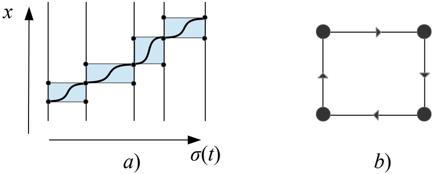
The Ricci scalar which is defined $R = g^{\mu\nu}R_{\mu\nu}$, i.e.,

$$R = -6\left(\frac{a^{\Delta\Delta}}{a} + \frac{(a^{\Delta})^2}{a^2} + \frac{k}{a^2}\right).$$

Conclusions

- Fibonacci numbers show analogies with inverted harmonic oscillator equation and FLRW
- both discrete and continous universe have the same dynamics
- discretization of time comes with the discretization of space
- Yang Mills theory: is space-time discrete or continuous?
- the theory is valid in comoving coordinates only; a covariant theory is needed





Thank you for your attention.