Stochastic processes in Astrophysics and Cosmology

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Stochastic processes in Astrophysics and Cosmology

Introduction to stochastic processes

A taste of what can be done in astrophysics and cosmology

Stochastic processes in Astrophysics and Cosmology

Stochastic Dissipative Stormer Problem

Introduction to stochastic processes

A taste of what can be done in astrophysics and cosmology

Harko & **Mocanu**, *The stochastic-dissipative Stormer problem – trajectories and radiation patterns*
In preparation

What does stochastic mean and why is it necessary?

(classic) Brownian Motion

Inability to keep track of experiments

Need an analytical tool to quantify this lack of knowledge

What does stochastic mean and why is it necessary?

Beyond classical BM

Inability to model nature with deterministic equations

Intrinsic randomness in nature

What does stochastic mean and why is it necessary?

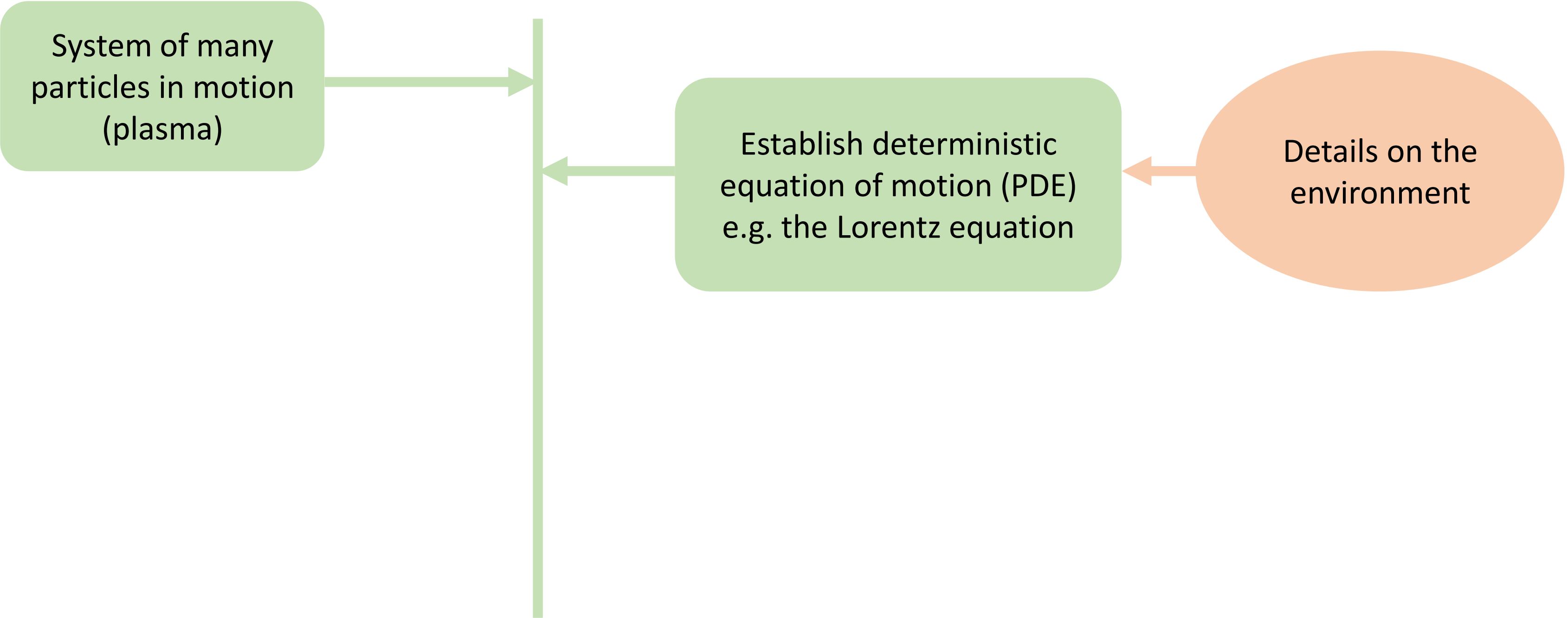
Beyond classical BM

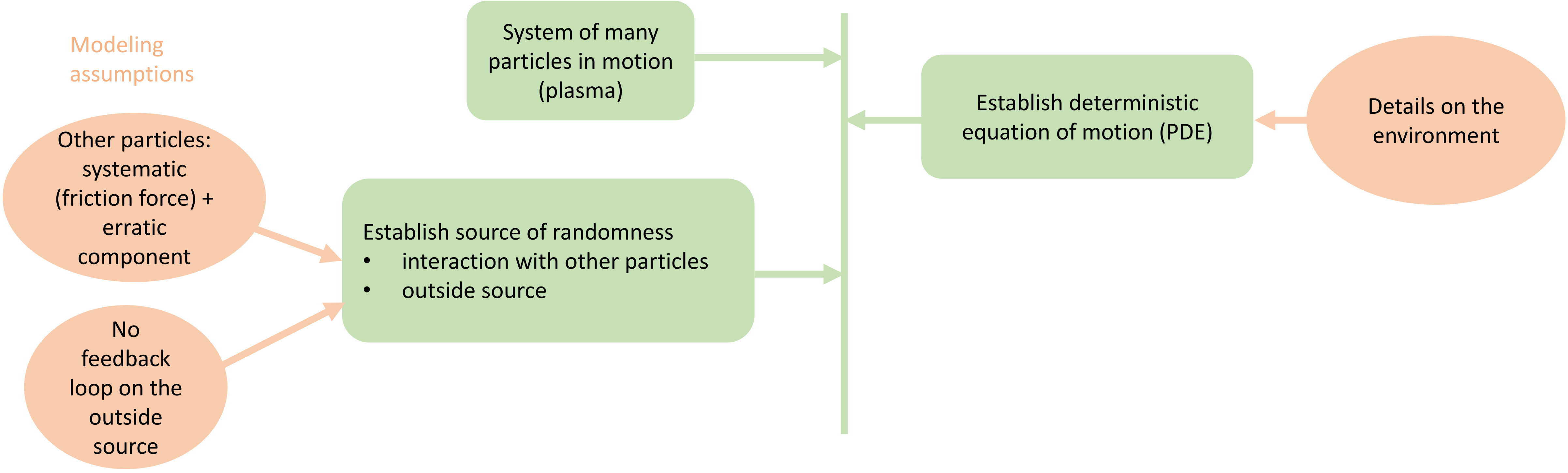
Sun related work

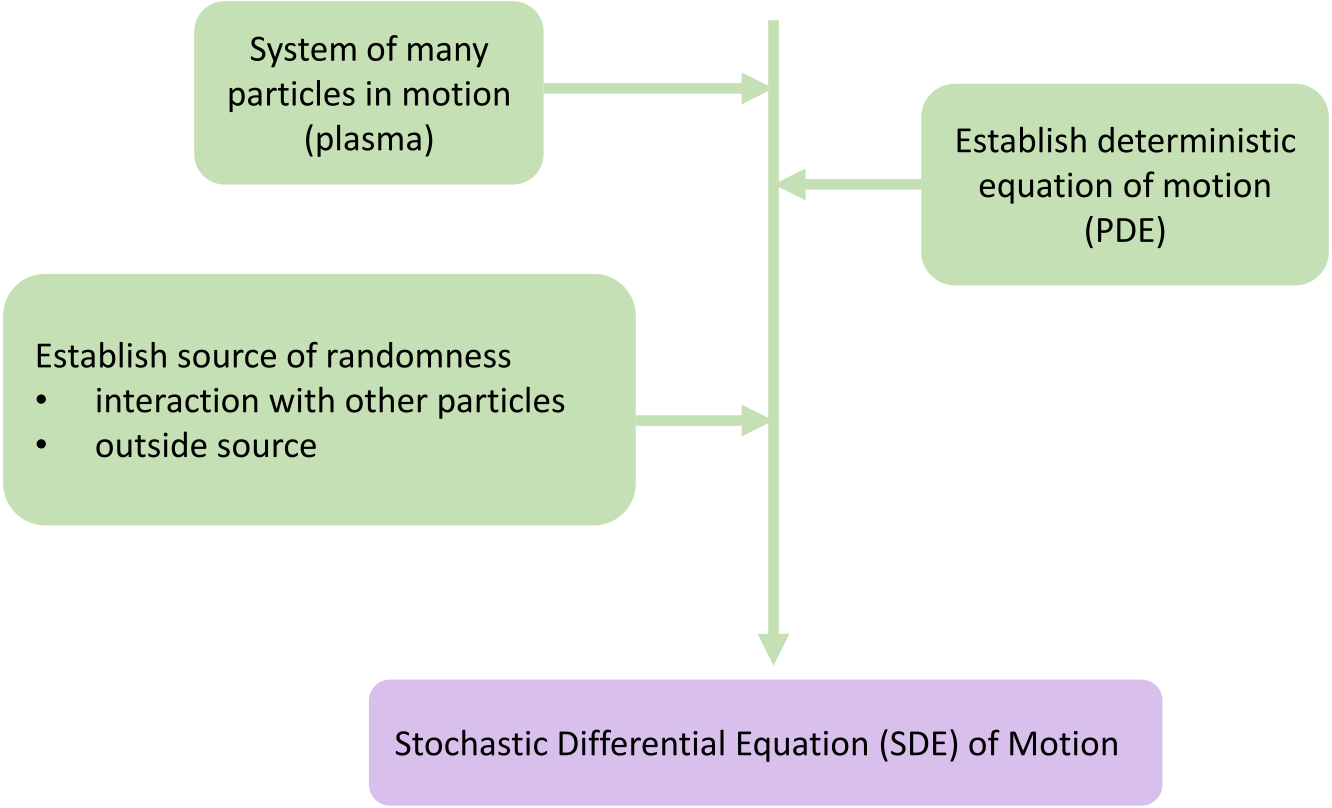
M.J. Aschwanden *A Statistical Fractal-Diffusive Avalanche Model of a Slowly-Driven Self-Organized Criticality System,* Astronomy & Astrophysics 539 (2011)

V. Chertoprud, B. Ioshpa, V. Obridko, *Fine-scale Stochastic Structure of Solar Magnetic Fields,* Astronomical Society of the Pacific Conference Series, 405, Solar Polarization, 205 (2009)

Modeling assumptions




Modeling assumptions
System of many particles in motion (plasma)
Establish deterministic equation of motion (PDE)
Details on the environment
Other particles: systematic (friction force) + erratic component
Establish source of randomness
• interaction with other particles
• outside source
No feedback loop on the outside source


System of many particles in motion (plasma)
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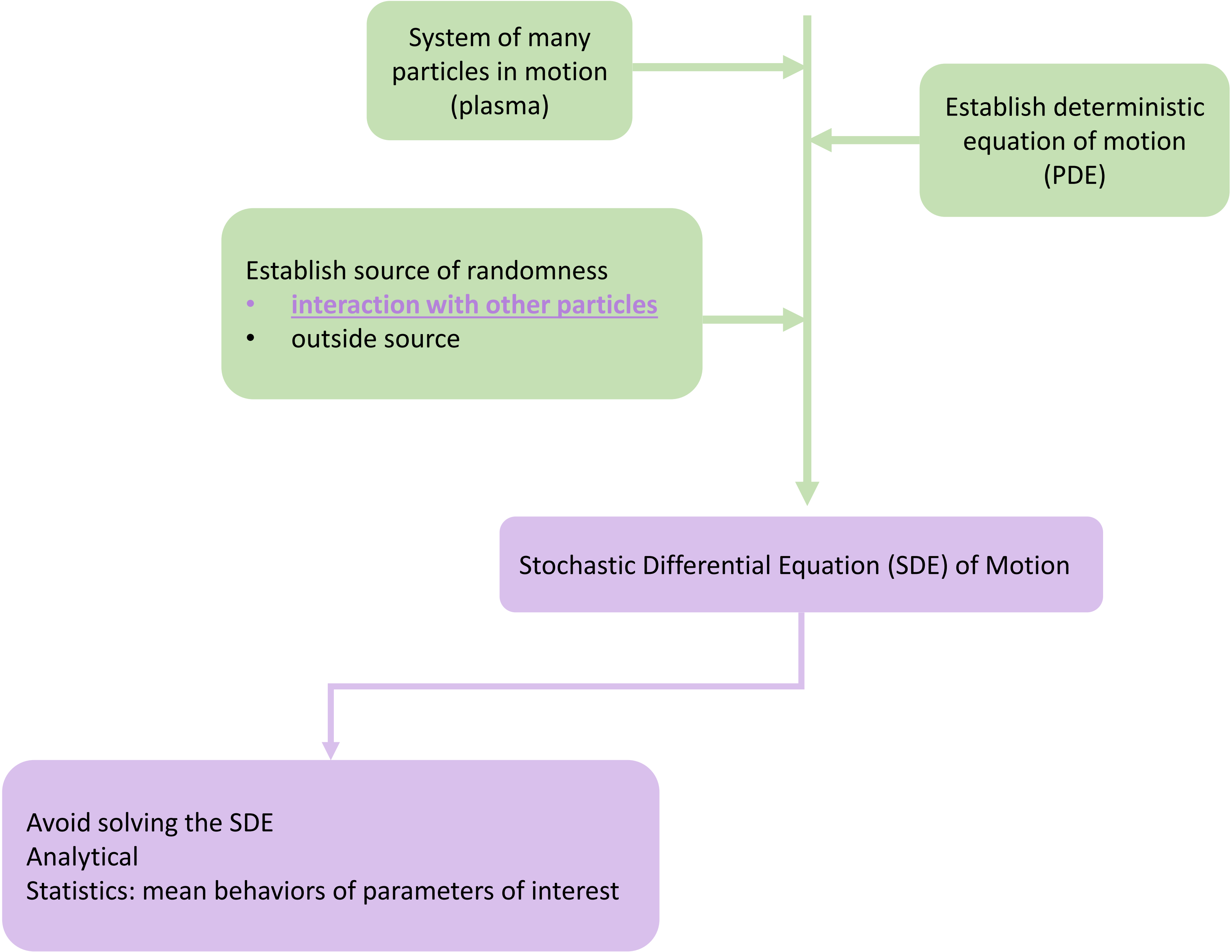
Stochastic Differential Equation (SDE) of Motion


System of many particles in motion (plasma)
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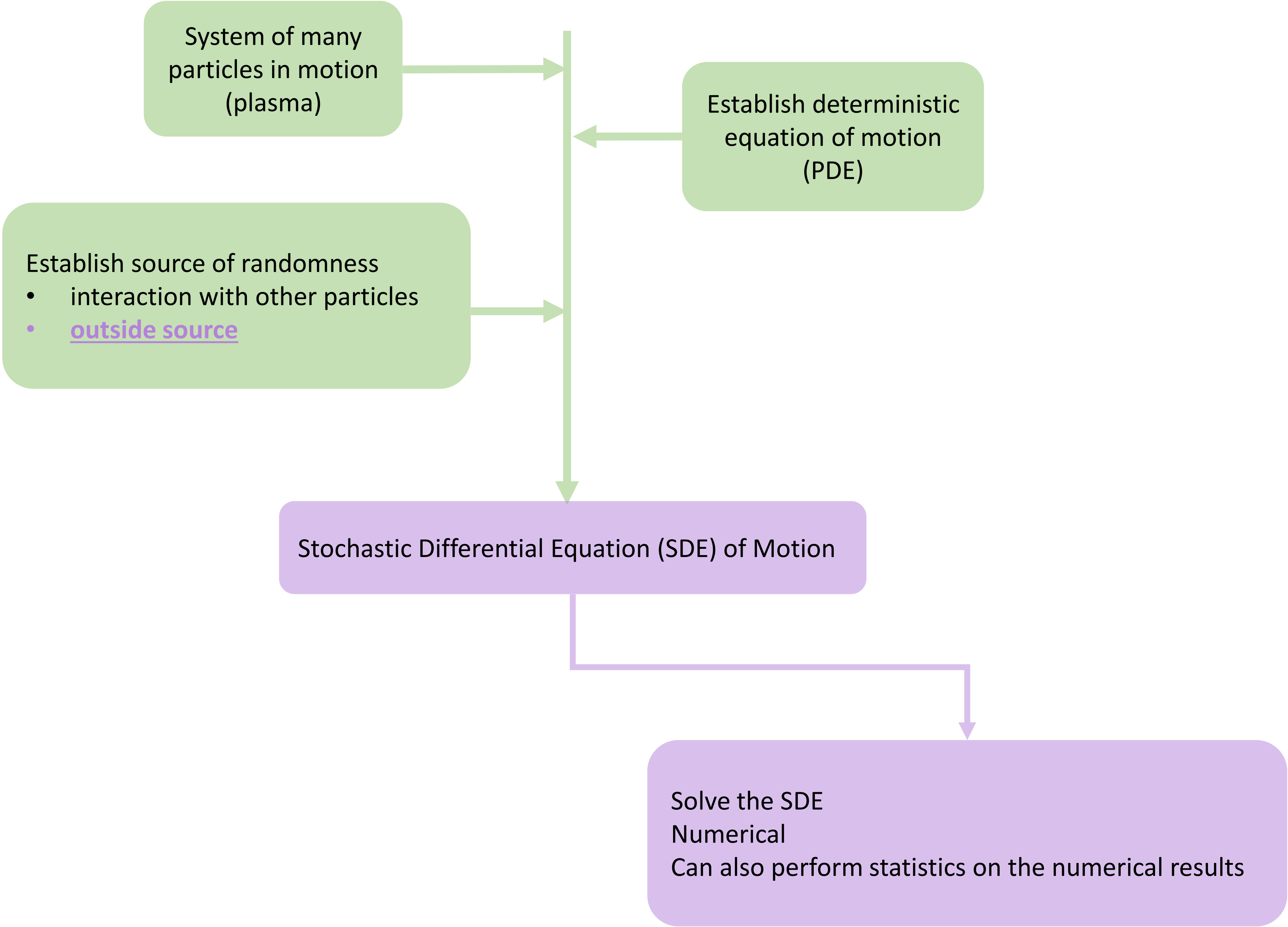
Stochastic Differential Equation (SDE) of Motion
Avoid solving the SDE
Analytical
Statistics: mean behaviors of parameters of interest


System of many particles in motion (plasma)
Establish deterministic equation of motion (PDE)
Establish source of randomness
• interaction with other particles
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Stochastic Differential Equation (SDE) of Motion
Solve the SDE
Numerical
Can also perform statistics on the numerical results

What actually happens to the equations?
We want to work our way up into describing the observed Brownian Motion of particles

Dynamical equation of the standard Wiener process W(t)

$$W(t+dt) - W(t) = \sqrt{dt} N_t^{t+dt}(0,1)$$

$$dW(t) = \sqrt{dt} N_t^{t+dt}(0,1)$$

$$\langle W(t) \rangle = 0$$

$$\langle W(t) W(t') \rangle = \min(t, t')$$

$$\langle dW(t) \rangle = 0$$

$$\langle dW(t) dW(t') \rangle = dt \delta_{tt'}$$

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We wish to describe nature

The hmmmm moment

$$V(t+dt) - V(t) = \sqrt{dt} N_t^{t+dt}(0,1)$$

Lemons, *An introduction to stochastic processes in physics*, Baltimore, Maryland, The Johns Hopkins University Press (2002)

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We wish to describe nature

Dynamical equation of the standard Brownian Motion
Langevin equation – equation of motion

$$V(t+dt) - V(t) = -\gamma V(t) dt + \beta^2 dW(t)$$

Equation for displacement

$$X(t+dt) - X(t) = V(t) dt$$

Einstein fluctuation-dissipation

$$\beta^2 \gamma = 2 k_B T$$

R. Kubo, *The fluctuation-dissipation theorem,* Reports on Progress in Physics, 29, 255 (1966)

What does stochastic mean and why is it necessary?

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Dynamical equation of the standard Brownian Motion
Langevin equation

$$V(t+dt) - V(t) = -\gamma V(t) dt + \beta^2 dW(t)$$

Mathematics: Ornstein–Uhlenbeck process

$$X(t+dt) - X(t) = -\gamma X(t) dt + \sigma dW(t)$$

Problems appearing in the mathematical description are addressed by Itô stochastic calculus

Øksendal, *Stochastic differential Equations, An introduction with applications, Springer, 2003*
Särkkä, *Lecture 2: Itô Calculus and Stochastic Differential Equations,* 2013, https://users.aalto.fi/~ssarkka/course_ox2013/pdf/handout2.pdf

What does stochastic mean and why is it necessary?

Mathematics: Ornstein–Uhlenbeck process

$$X(t+dt) - X(t) = -\gamma X(t)dt + \sigma dW(t)$$

Use this framework for other processes that can be described by the OU process (not necessarily velocity of a Brownian Particle)

The tevolution of the stochastic process X(t) is a succession of random variables:

X(t) and X(t + dt) are two different random variables as they have the probability densities p(x, t) and p(x, t+dt) respectively

These two different random variables are related by a dynamical equation,

$$X(t+dt) - X(t) = F[X(t), t]$$

stochastic propagator

This is the update form of the differential equation $\frac{dX(t)}{dt} = \frac{dF}{dt}$

But the stochastic process X(t) is not smooth so the derivative expression is technically incorrect

Name	Probability density	SDE	Mean and Variance	Physics
Gaussian white noise $\xi(t)$	$\mathcal{N}(\mu,\sigma)$		standard: $\mu = 0$, $\sigma = 1$	
Wiener process $W(t)$	$\mathcal{N}(\mu,\sigma(t))$	$W(t+dt)-W(t) = \sqrt{dt}N_t^{t+dt}(0,1)$	standard: $\mu = 0$, $\sigma = t$	
Wiener increment $dW(t) = W(t+dt) - W(t)$	$\mathcal{N}(\mu,\sigma)$	$dW(t) = \sqrt{dt}N_t^{t+dt}(0,1)$	standard: $\mu = 0$, $\sigma = dt$	
OU process $X(t)$		$X(t+dt) - X(t) = \mu X(t)dt + \sigma dW(t)$		mathematical description
Brownian motion $V(t)$	$\mathcal{N}(\mu(t),\sigma(t))$	$V(t+dt) - V(t) = -\gamma V(t)dt + \sqrt{\beta^2 dt}N_t^{t+dt}(0,1)$	$\mu(t)$ in Eq. (4.11), $\sigma(t)$ in Eq. (4.13)	as found in physics. Velocity of a particle in erratic motion in an environment at temperature T, $\frac{\beta^2}{2\gamma} = \frac{k_B T}{2}$
Integral of Brownian motion $X(t)$	$\mathcal{N}(\mu(t),\sigma(t))$	$dX(t) = V(t)dt$	$\mu(t)$ in Eq. (4.15), $\sigma(t)$ in Eq. (4.17)	
Geometric Brownian motion $X(t)$		$X(t+dt)-X(t) = X(t)\left(\mu + \frac{\sigma^2}{2}dt + \sigma X(t)dW(t)\right)$		

Overview of some stochastic processes and their properties

System of many particles in motion (plasma)

Establish deterministic equation of motion (PDE)

Establish source of randomness

- interaction with other particles
- ~~outside source~~

Stochastic Differential Equation (SDE) of Motion

Brownian Motion

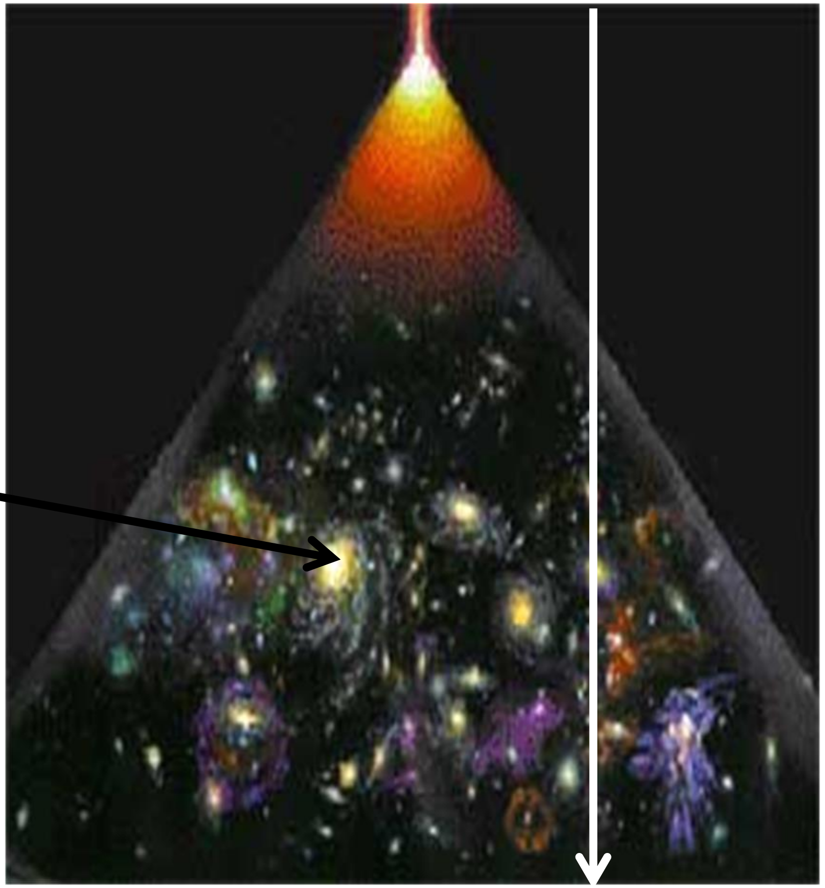
$$V(t+dt)-V(t)=-\gamma V(t)dt+\beta^2 dW(t)$$

$$X(t+dt)-X(t)=V(t)dt$$

$$\beta^2\gamma=2k_B T$$

Build on this to find trajectory and radiation of charged particles

Big Bang



Stochastic oscillations of general relativistic discs

Harko, **Mocanu**, *MNRAS*, 421, 3102–3110, 2012

Ref.	Spectral band	Object	Comments
Ioshpa *et al.* (2007)	Magnetic Doppler Imager (MDI)	The Sun	observation: small scale stochastic structure of the solar magnetic field, fractal index;
MacLeod *et al.* (2010)	optical	9000 quasars	observation: the smooth PSD of LC on thermal timescale suggests a chaotic or stochastic origin for variability;
Carini *et al.* (2011)	optical	0716+714	observation: the observed variations are determined by the size of the turbulent regions (and not the electron cooling or acceleration timescales intrinsic to those timescales); LCs do not show any sign of periodicity; conclude that the observed variability is a result of a fractional noise process, consistent with the variations arising from a turbulent process;
Wagner & Witzel (1995)	optical	0954+658	observation: the outbursts are self similar, with exponential slopes of constant e-folding time [1]
Hoshino & Takeshima (1993)	X-Ray	X-Ray Binaries, Active Galactic Nuclei	observation: can be well interpreted as Self Organized state of plasma turbulence in the accreting plasmas;
Azarnia *et al.* (2005)	R band	S5 0716+714	observation: analysis with Fourier transform show that a fractional noise process is responsible for the variations;
Leung *et al.* (2011a)	optic and X-Ray	Bl Lac	observation: calculate the fractal dimension and found that the source exhibits an almost pure random walk behaviour.
Uttley *et al.* (2005)	X-Ray	small BHs	observation: the rms-flux relation says that the underlying variability process must be multiplicative

Ref.	Spectral band	Object	Comments
Malzac *et al.* (2004)	X-Ray	XTE J1118+48	observation: $rms \propto B$
Ohsuga *et al.* (2005)	—	Sgr*	simulation: MC radiative transfer simulation in 3D MHD show that the MHD flow is intrinsically time varying and exhibits fractal structure;
Kawaguchi *et al.* (2000); Takeuki *et al.* (1995); Kawaguchi *et al.* (1998)	—	—	theory and simulation: fractal magnetic fields produce $f^{-\beta}$ fluctuations;
Kelly *et al.* (2009)	optical	100 quasars	observation and simulation: LC on thermal timescales result from fluctuations that are driven by an underlying stochastic process, such as a turbulent magnetic field;
Kelly *et al.* (2009)	X-Ray	small mass (XRBs)	observation and theory: the black hole is a Gaussian stochastic process in the logarithm of the flux;
Janiuk & Czerny (2007)	X-Ray	—	simulation: variability is driven by variations in a magnetic field with the magnetic field density being modelled as an AR(1) process; $dX(t) = -\frac{1}{\tau}X(t)dt + \sigma\sqrt{dt}\epsilon(t) + bdt,$ (B.1) where $b, \tau, \sigma > 0$, $\epsilon(t) \in N(0,1)$, making $X(t) \in N\left(b\tau, \frac{\tau\sigma^2}{2}\right)$. If $X(t)$ is a CAR(1) process theory says that $P_X(f) = \frac{2\sigma^2\tau^2}{1+(2\pi\tau f)^2}.$ (B.2)
Kawaguchi *et al.* (2000)	—	—	simulation: examine the 3d MHD simulation data of Machida *et al.* (2000) and find that a magnetized accretion disk exhibits both $f^{-\beta}$ fluctuations and a fractal magnetic field structure, with the fractal dimension of ≈ 1.9;

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Ioshpa *et al.* (2007)	Magnetic Doppler Imager (MDI)	The Sun	observation: small scale s… structure of the solar magn… fractal index;
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			…the black …tic process …ux;
			…driven by …field with …sity being …ocess;
			…$\overline{u}x(t)+bdt$, (B.1) $\in \mathcal{N}(0,1)$, …). If $X(t)$ …y says that
			… (B.2)
			… 3d MHD …uida *et al.* …nagnetized …both $f^{-\beta}$ fluctuations and a fractal magnetic field structure, with the fractal dimension of ≈ 1.9;


The Daily W…

The magnetic field did it

Incredible as it may seem, it has been confirmed that a large martian invasion fleet has landed on earth tonight.

First vessels were sighted over Great Britain, Denmark and Norway already in the late evening from where, as further reports indicate, the fleet headed towards the North Pole and Santa Claus was taken hostage by the invaders.

Afterwards they split apart in order to approach most major cities around the earth. The streets filled as thousands fled their homes, many only wearing their pajamas…

System of many particles in motion (plasma)

Establish deterministic equation of motion (PDE)


The Daily

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Establish source of randomness

- interaction with other particles
- outside source


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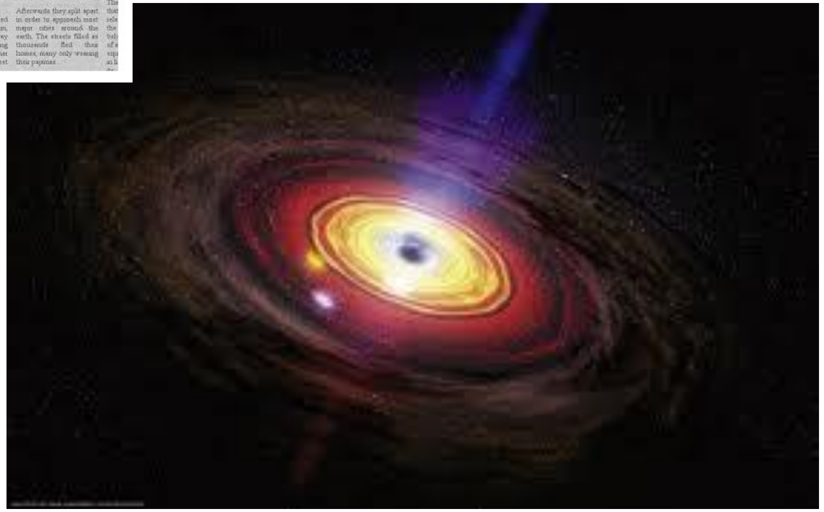
Stochastic Differential Equation (SDE) of Motion



Explanation of Active Galactic Nuclei observational features

Modelling tools for accretion disk dynamics

Magnetized disk and fractal perturbation

Mocanu, Marcu, *Astronomische Nachrichten*, 333, pages 166–173, 2012
Mocanu, Sándor, *Astrophys. Space Sci.*, 279, pages 147-153, 2012
Mocanu, Grumiller, *Phys. Rev. D*, 85, 105022, 2012
Mocanu, Pardi, Magyar, Marcu, *MNRAS,* 439, 3790 – 3797, 2014
Harko, Leung**, Mocanu**, *European Physical Journal C*, (2014), 74:2900
Leung, **Mocanu**, Harko, *Journal of Astrophysics and Astronomy* (2014) 35, 449-452
Danila, Harko, **Mocanu**, *MNRAS*, 453, 3, 2982–2991, 2015
Harko, **Mocanu**, Stroia, *Astrophysics and Space Science* 357 (2015), 1-9
Danila, Marcu, **Mocanu**, *Research in Astronomy and Astrophysics*, 15 (2015), 3, 327-332
Harko, **Mocanu,** *European Journal of Physics C*, 76, 160 (2016)

<u>Brownian Motion in an environment in which a deterministic Magnetic field exists, but it is not time dependent</u>

Theoretical and numerical update of the model


The Daily
The magnetic field did it

Mocanu, *Romanian Astronomical Journal*, 29, 1, 41-57, 2019

Mocanu, *Romanian Reports in Physics*, 72, 1, 105, 2020

Mocanu, *IEEE Transactions on Plasma Science*, 99, 1:9, 2021

Main result: Trajectories of Charged Particles Undergoing Brownian Motion in a Time-Dependent Magnetic Field (deterministic)

In the limit of constant magnetic field, recovers the results in Lemons & Kaufman, *Brownian Motion of a Charged Particle in a Magnetic Field,* IEEE Transactions on Plasma Science 27, 5, 1288 (1999)

In preparation: **Mocanu**, *Trajectories of Charged Particles Undergoing Brownian Motion in a Time-Dependent Magnetic Field* (stochastic)

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The Daily W

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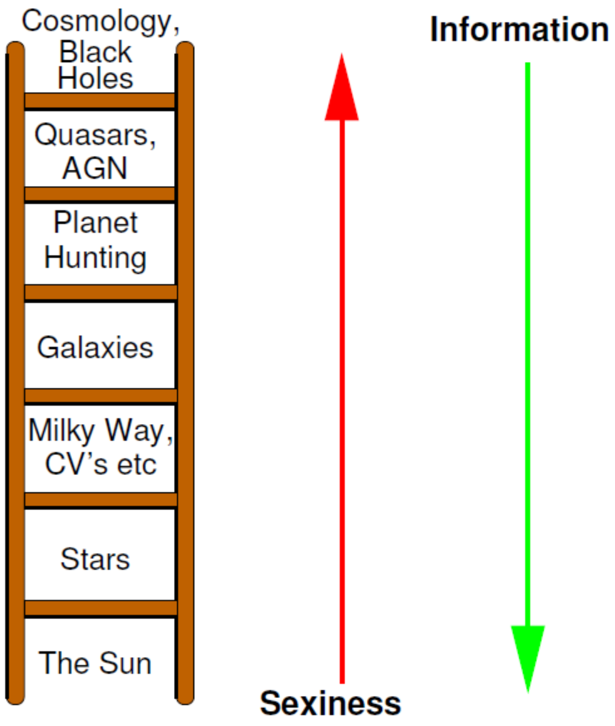
Stochastic Differential Equation (SDE) of Motion

Observational data

The Cosmic Sexiness Ladder



J.J. Drake

Harko & Mocanu, *The stochastic-dissipative Stormer problem – trajectories and radiation patterns*
In preparation

Charged particle in motion:	classical and non-relativistic Stormer (motion in a dipole magnetic field) dissipation Brownian Motion – Stochastic differential equation of motion
Results:	dynamical behavior radiation (by different mediation procedures) escape rate
Tools employed:	analytic numerical solve SDE (Milstein Scheme) apply the 0-1 test for chaos

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Applications to observations: ?

Lorentz-Langevin equation

$$m\frac{d^2\boldsymbol{r}}{dt^2} = q\boldsymbol{v}\times\boldsymbol{B} - \gamma m\boldsymbol{v} + m\boldsymbol{f}^s$$

$$\langle f_i^s(t)\rangle = 0$$

$$\left\langle f_i^s(t_1) f_j^s(t_2)\right\rangle = \frac{a}{m^2}\delta_{ij}\delta(t_1 - t_2)$$

Magnetic dipole created by a current loop in the *xOy* plane

$$\boldsymbol{A} = \frac{M_z 1}{r^3}(-y\widehat{\boldsymbol{x}} + x\widehat{\boldsymbol{y}}), \boldsymbol{B} = \nabla\times\boldsymbol{A}$$

$$\frac{d^2x}{dt^2} = \frac{3\alpha z}{r^5}(\dot{y}z - \dot{z}y) - \alpha\frac{1}{r^3}\dot{y} - \gamma\dot{x} + f_x^s$$

$$\frac{d^2y}{dt^2} = -\frac{3\alpha z}{r^5}(\dot{x}z - \dot{z}x) - \alpha\frac{1}{r^3}\dot{x} - \gamma\dot{y} + f_y^s$$

$$\frac{d^2z}{dt^2} = \frac{3\alpha z}{r^5}(\dot{x}y - \dot{y}x) - \gamma\dot{z} + f_z^s$$

$$\alpha = \frac{qM_z}{m}$$

Stochastic processes are not smooth, so the derivative notation is understood to be formal

Make equations dimensionless

$$X_i = \frac{x_i}{r_0}$$

$$\tau = \beta t$$

$$\frac{d^2X}{d\tau^2} = 3\frac{Z}{R^5}\left(\dot{Y}Z - \dot{Z}Y\right) - \frac{1}{R^3}\dot{Y} - \Gamma\dot{X} + \Phi_X^s$$

$$\frac{d^2Y}{d\tau^2} = -3\frac{Z}{R^5}\left(\dot{X}Z - \dot{Z}X\right) - \frac{1}{R^3}\dot{X} - \Gamma\dot{Y} + \Phi_Y^s$$

$$\frac{d^2Z}{d\tau^2} = 3\frac{Z}{R^5}\left(\dot{X}Y - \dot{Y}X\right) - \Gamma\dot{Z} + \Phi_Z^s$$

$$\dot{X}_i = \frac{dX_i}{d\tau}$$

Make equations dimensionless

$$X_i = \frac{x_i}{r_0}$$

$$\tau = \beta t$$

$$\frac{d^2X}{d\tau^2} = 3\frac{Z}{R^5}\left(\dot{Y}Z - \dot{Z}Y\right) - \frac{1}{R^3}\dot{Y} - \Gamma\dot{X} + \Phi_X^s$$

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$$\frac{d^2Z}{d\tau^2} = 3\frac{Z}{R^5}\left(\dot{X}Y - \dot{Y}X\right) - \Gamma\dot{Z} + \Phi_Z^s$$

$$\dot{X}_i = \frac{dX_i}{d\tau}$$

Solve coupled system of dimensionless SDEs with Millstein scheme

Giles, *Advanced Monte Carlo Methods*, Oxford University Mathematical Institute, https://people.maths.ox.ac.uk/gilesm/talks/giles_module6.pdf

$$d\boldsymbol{S}(\tau) = \boldsymbol{A}(\boldsymbol{S})d\tau + B(\boldsymbol{S})d\boldsymbol{W}(\tau)$$

$$d\boldsymbol{S}(\tau) = (dV_X, dX, dV_Y, dY, dV_Z, dZ)$$

$$A_1(\boldsymbol{S}) = 3\frac{Z}{R^5}(V_Y Z - V_Z Y) - \frac{1}{R^3}V_Y - \Gamma V_X \qquad A_2(\boldsymbol{S}) = V_X$$

$$A_3(\boldsymbol{S}) = -3\frac{Z}{R^5}(V_X Z - V_Z X) - \frac{1}{R^3}V_X - \Gamma V_Y \qquad A_4(\boldsymbol{S}) = V_Y$$

$$A_5(\boldsymbol{S}) = 3\frac{Z}{R^5}(V_X Y - V_Y X) - \Gamma V_Z \qquad A_6(\boldsymbol{S}) = V_Z$$

$B(\boldsymbol{S})$ $\quad 6 \times 6\ matrix$

$$B_{11} = B_{33} = B_{55} = 1; B_{other} = 0$$

$$d\boldsymbol{W}(\tau) = \left(dW_X(\tau), 0, dW_Y(\tau), 0, dW_Z(\tau)\right)$$

Solve coupled system of SDEs with Millstein scheme

$$d\boldsymbol{S}(\tau) = \boldsymbol{A}(\boldsymbol{S})d\tau + B(\boldsymbol{S})d\boldsymbol{W}(\tau)$$

$$d\boldsymbol{S}(\tau) = (dV_X, dX, dV_Y, dY, dV_Z, dZ)$$

$$\tau = nh$$

Dimensionless velocity and position

$$S_i(n+1) = S_i(n) + A_i\big(\boldsymbol{S}(n)\big)h + B_{ij}(n)dW_i(n)\delta_{ij}$$

Dimensionless acceleration

$$a_i = A_i(\boldsymbol{S}) + \sigma_{\Phi i}\, N_{2i}$$

N_{2i} −no. drawn at each timestep and for each vector component from a standard unit normal

Classical Stormer Problem (CSP)

$$\frac{d^2X}{d\tau^2} = 3\frac{Z}{R^5}\left(\dot{Y}Z - \dot{Z}Y\right) - \frac{1}{R^3}\dot{Y}$$

$$\frac{d^2Y}{d\tau^2} = -3\frac{Z}{R^5}\left(\dot{X}Z - \dot{Z}X\right) - \frac{1}{R^3}\dot{X}$$

$$\frac{d^2Z}{d\tau^2} = 3\frac{Z}{R^5}\left(\dot{X}Y - \dot{Y}X\right)$$

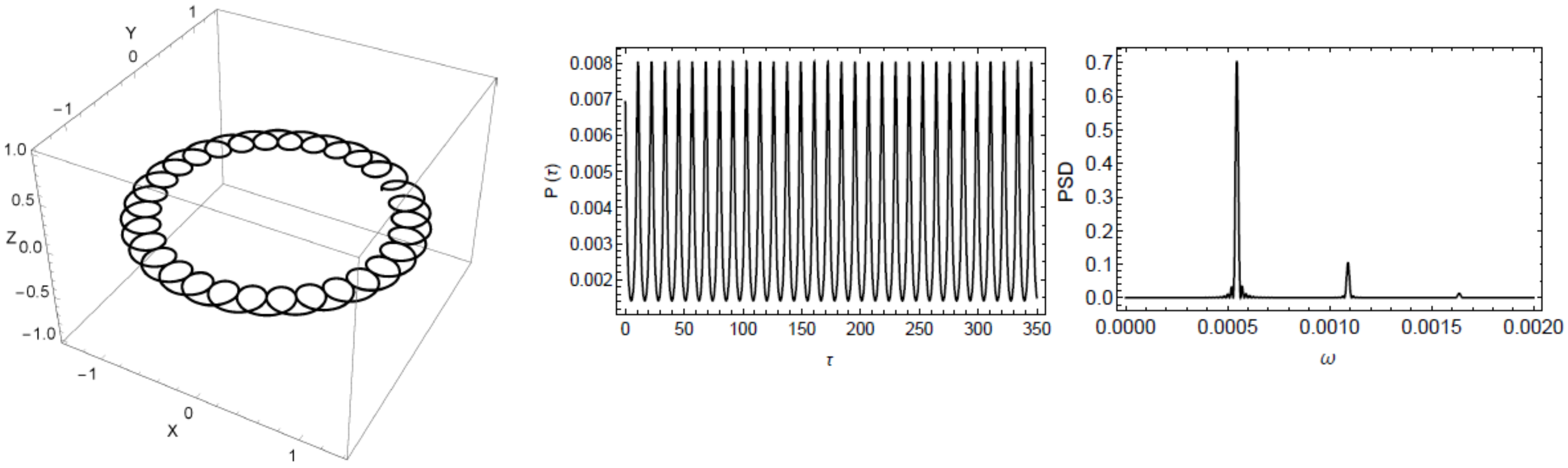

Figure 1. Trajectory, emitted power, and PSD of the emitted power for $X_0 = 0.7$, $Y_0 = 0.8$, $Z_0 = 0$, $V_{x0} = 0.1$, $V_{y0} = 0$, $V_{z0} = 0$, $h = 0.001$, $L = 350000$, the PSD was obtained by sampling the $P(\tau)$ at timesteps of $h = 0.001$, making the length of the array $L = 350000$. The PSD is applied to the same space as for the other cases and is thus comparable directly to the other PSDs.

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Classical Stormer Problem

$$\frac{d^2X}{d\tau^2} = 3\frac{Z}{R^5}\left(\dot{Y}Z - \dot{Z}Y\right) - \frac{1}{R^3}\dot{Y}$$

$$\frac{d^2Y}{d\tau^2} = -3\frac{Z}{R^5}\left(\dot{X}Z - \dot{Z}X\right) - \frac{1}{R^3}\dot{X}$$

$$\frac{d^2Z}{d\tau^2} = 3\frac{Z}{R^5}\left(\dot{X}Y - \dot{Y}X\right)$$

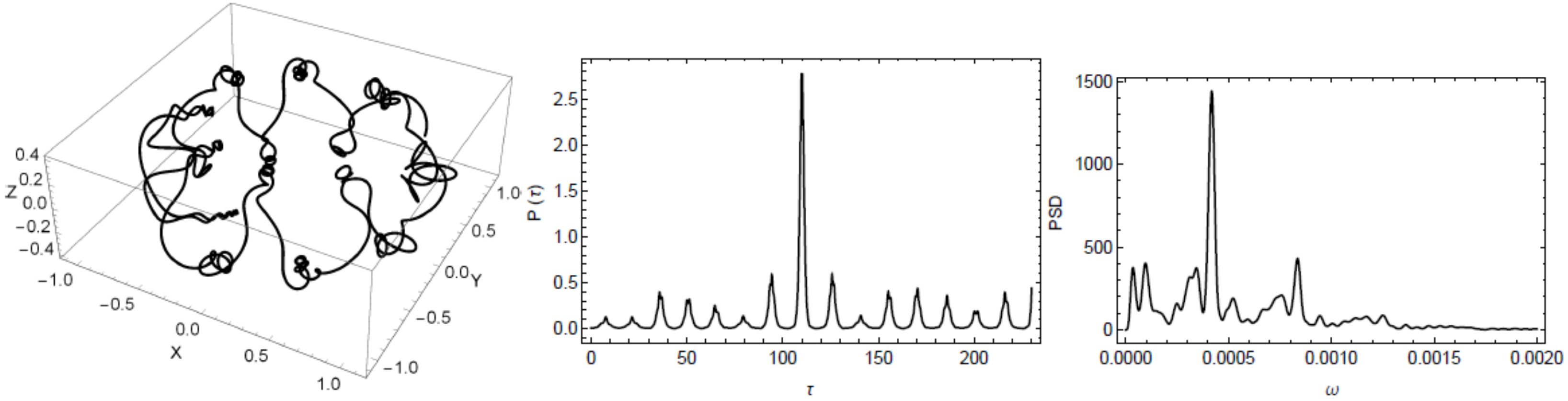

Figure 2. Trajectory, emitted power, and PSD of the emitted power for $X_0 = 0.7$, $Y_0 = 0.8$, $Z_0 = 0$, $V_{x0} = 0$, $V_{y0} = 0.1$, $V_{z0} = 0.1$, $h = 0.001$, $L = 230000$

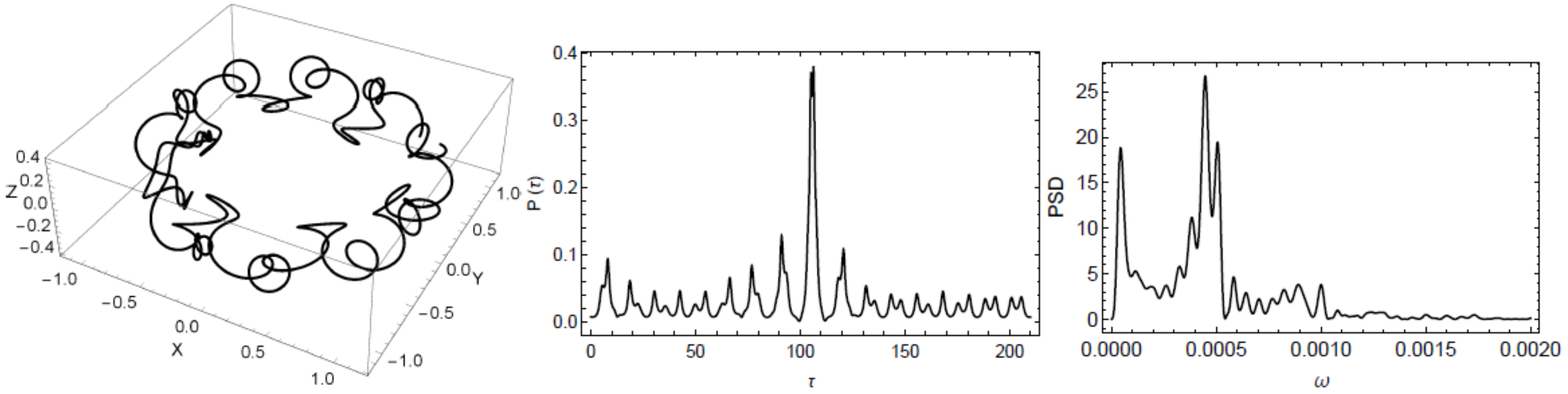

Figure 3. Trajectory, emitted power, and PSD of the emitted power for $X_0 = 0.7$, $Y_0 = 0.8$, $Z_0 = 0$, $V_{x0} = 0.01$, $V_{y0} = 0.1$, $V_{z0} = 0.1$, $h = 0.001$, $L = 210000$

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Confirm Chaos in the Classical Stormer Problem

Gottwald & Melbourne, *A new test for chaos in deterministic systems,* Proceedings of the Royal Society A, 460, 2042 (2004)

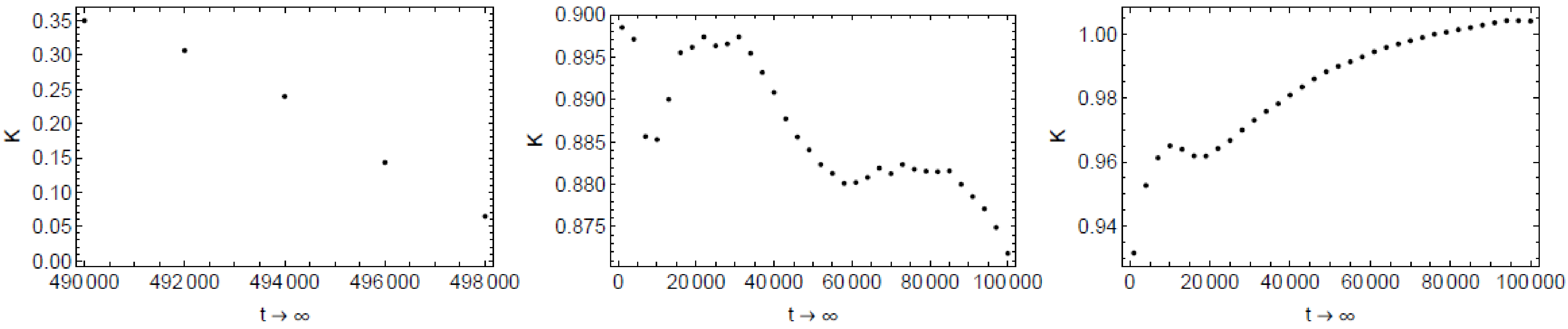

Figure 4. Results of the 0-1 chaos test for $T = 500000$ and different values of t for the three CSP cases discussed to far.

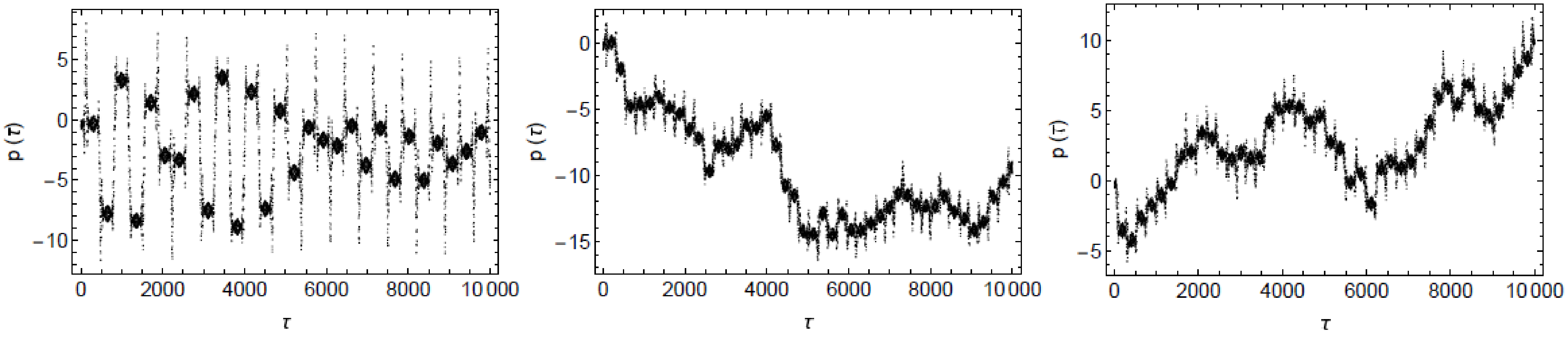

Figure 5. The function $p(\tau)$ for the CSP problem; left: trajectory from Figure 1, middle: trajectory from Figure 2, right: trajectory from Figure 3

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Classical Stormer Problem with Friction

$$\frac{d^2X}{d\tau^2} = 3\frac{Z}{R^5}\left(\dot{Y}Z - \dot{Z}Y\right) - \frac{1}{R^3}\dot{Y} - \Gamma\dot{X}$$

$$\frac{d^2Y}{d\tau^2} = -3\frac{Z}{R^5}\left(\dot{X}Z - \dot{Z}X\right) - \frac{1}{R^3}\dot{X} - \Gamma\dot{Y}$$

$$\frac{d^2Z}{d\tau^2} = 3\frac{Z}{R^5}\left(\dot{X}Y - \dot{Y}X\right) - \Gamma\dot{Z}$$

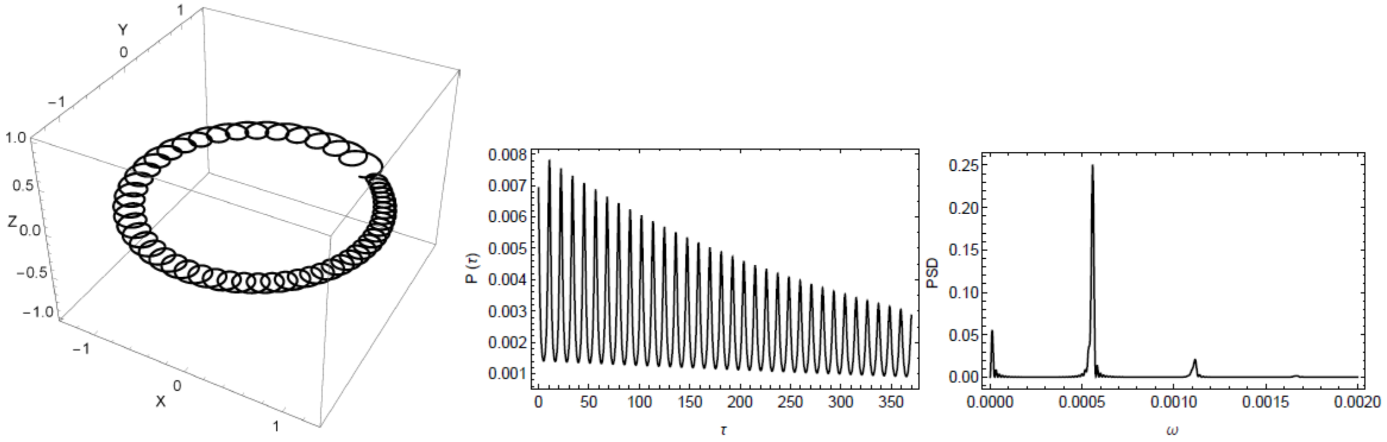

Figure 8. Trajectory, emitted power, and PSD of the emitted power in the Dissipative Störmer Problem for $X_0 = 0.7$, $Y_0 = 0.8$, $Z_0 = 0$, $V_{x0} = 0.1$, $V_{y0} = 0$, $V_{z0} = 0$, $h = 0.001$, $L = 580000$ and $\Gamma = 10^{-3}$.

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Classical Stormer Problem with Friction

$$\frac{d^2X}{d\tau^2} = 3\frac{Z}{R^5}\left(\dot{Y}Z - \dot{Z}Y\right) - \frac{1}{R^3}\dot{Y} - \Gamma\dot{X}$$

$$\frac{d^2Y}{d\tau^2} = -3\frac{Z}{R^5}\left(\dot{X}Z - \dot{Z}X\right) - \frac{1}{R^3}\dot{X} - \Gamma\dot{Y}$$

$$\frac{d^2Z}{d\tau^2} = 3\frac{Z}{R^5}\left(\dot{X}Y - \dot{Y}X\right) - \Gamma\dot{Z}$$

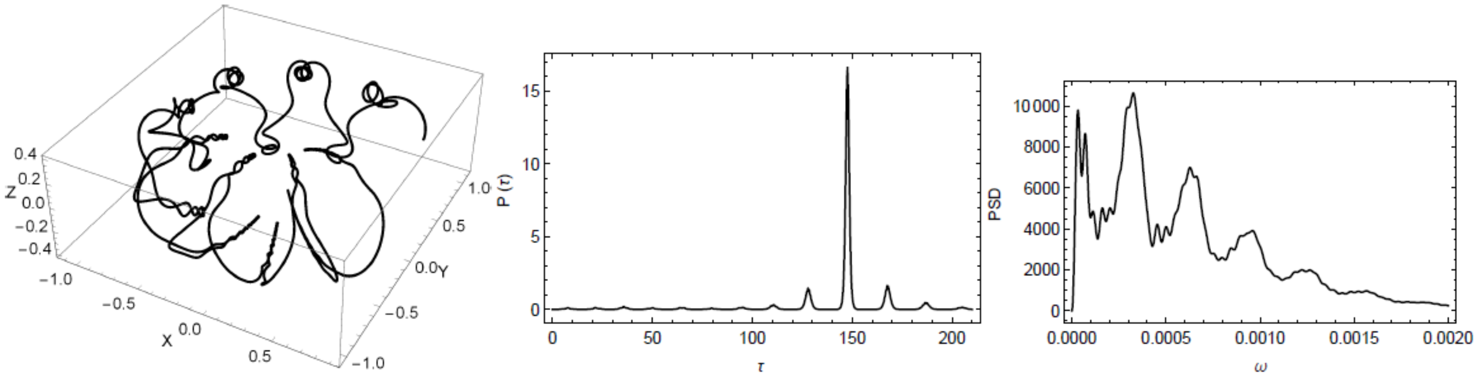

Figure 12. Trajectory, emitted power, and PSD of the emitted power in the Dissipative Störmer Problem for $X_0 = 0.7$, $Y_0 = 0.8$, $Z_0 = 0$, $V_{x0} = 0$, $V_{y0} = 0.1$, $V_{z0} = 0.1$, $h = 0.001$, $L = 260000$ and $\Gamma = 10^{-3}$.

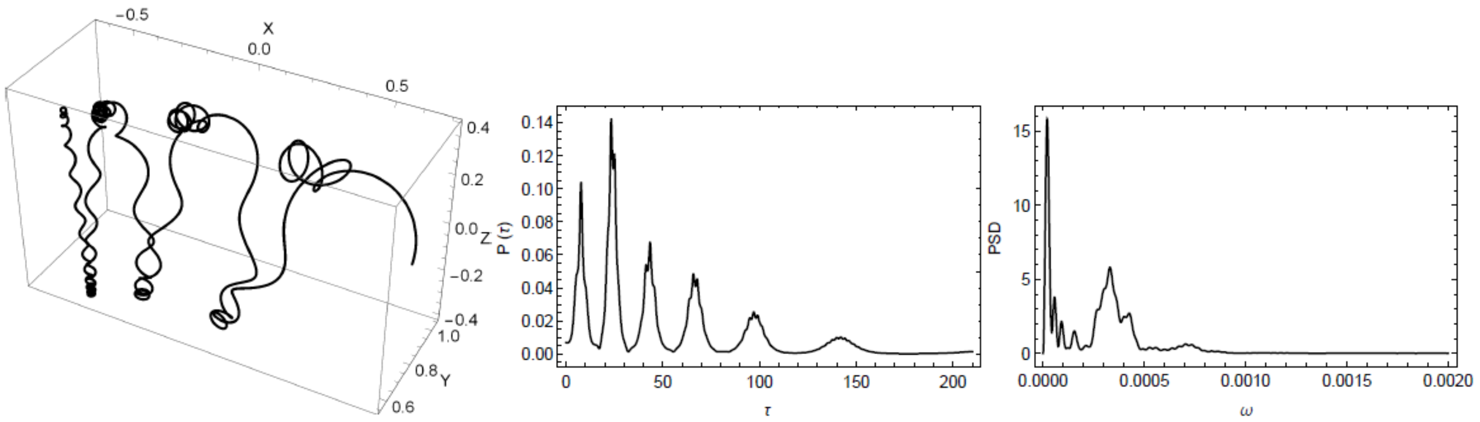

Figure 13. Trajectory, radiation and PSD for $X_0 = 0.7$, $Y_0 = 0.8$, $Z_0 = 0$, $V_{x0} = 0$, $V_{y0} = 0.1$, $V_{z0} = 0.1$, $h = 0.001$, $L = 200000$ and $\Gamma = 10^{-2}$.

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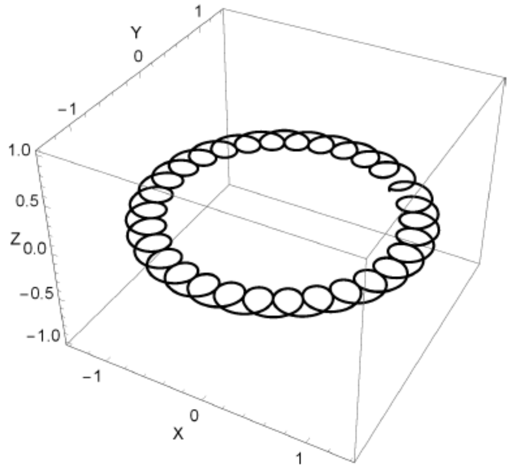
Brownian Motion in the Classical Stormer Problem

Escape appearing in otherwise closed orbits (non-chaotic ICs)

CSP



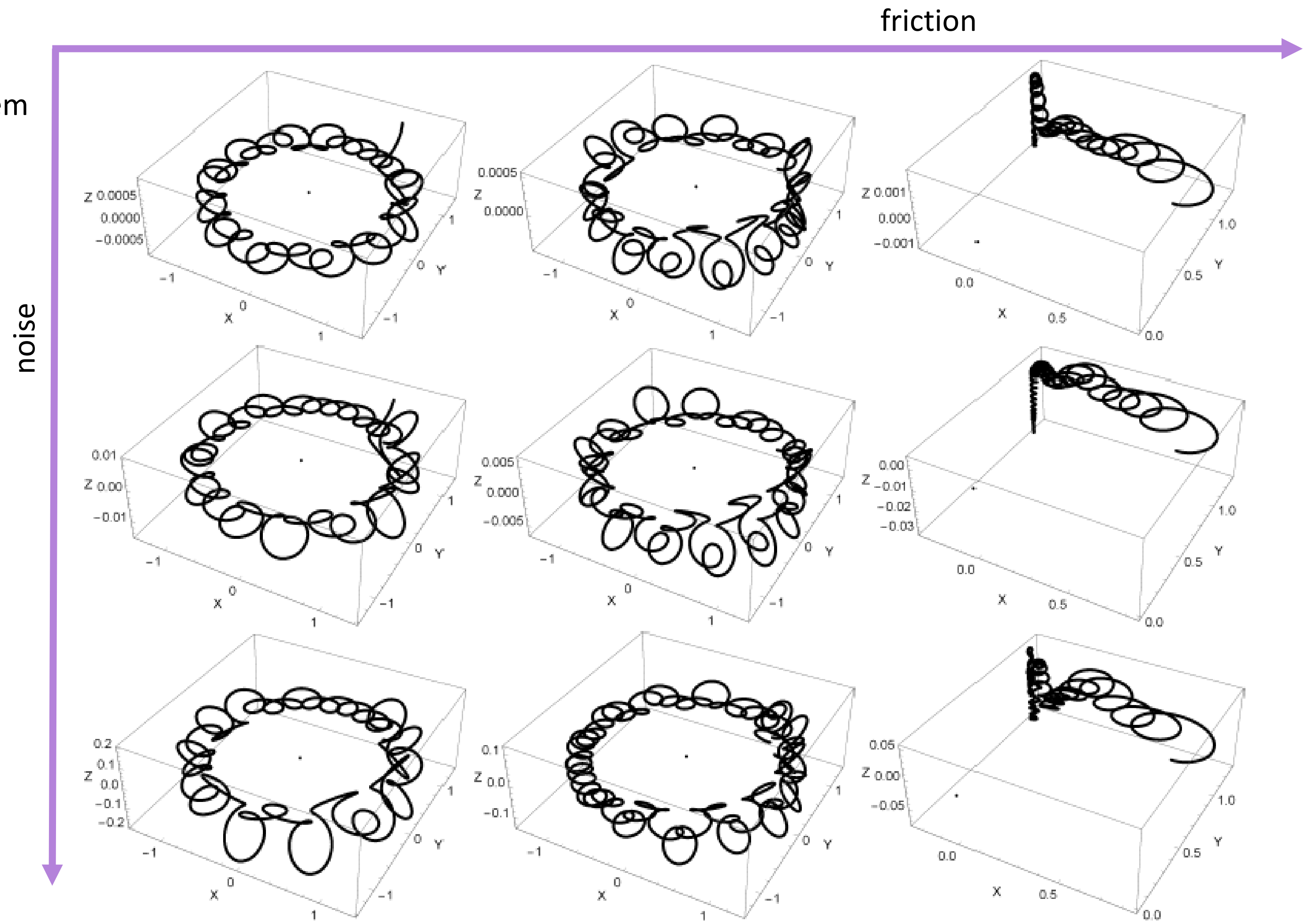


Figure 19. Brownian Motion in the Störmer Problem: trajectories for $X_0 = 0.7$, $Y_0 = 0.8$, $Z_0 = 0$, $V_{x0} = 0.1$, $V_{y0} = 0$, $V_{z0} = 0$, with upper row: $\sigma_S = 10^{-7}$, middle row $\sigma_S = 10^{-6}$ and lower row $\sigma_S = 10^{-5}$; for all rows, from left to right $\Gamma \in \{10^{-4}, 10^{-3}, 10^{-2}\}$

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Brownian Motion in the Classical Stormer Problem

Escape appearing in otherwise closed orbits
(chaotic ICs)

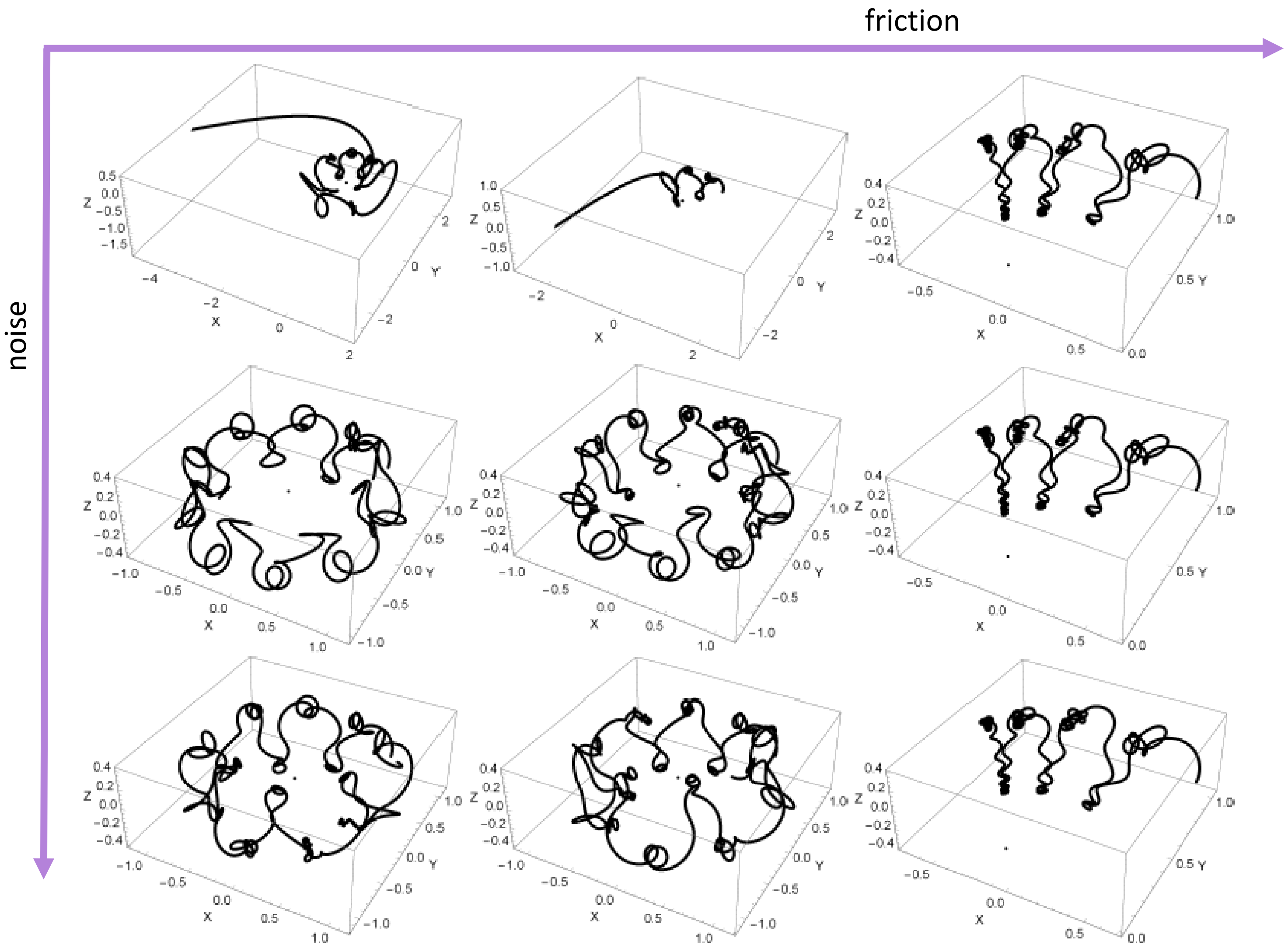

Figure 20. Brownian Motion in the Störmer Problem: trajectories for $X_0 = 0.7$, $Y_0 = 0.8$, $Z_0 = 0$, $V_{x0} = 0$, $V_{y0} = 0.1$, $V_{z0} = 0.1$, with upper row: $\sigma_S = 10^{-7}$, middle row $\sigma_S = 10^{-6}$ and lower row $\sigma_S = 10^{-5}$; for all rows, from left to right $\Gamma \in \{10^{-4}, 10^{-3}, 10^{-2}\}$

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Brownian Motion in the Classical Stormer Problem

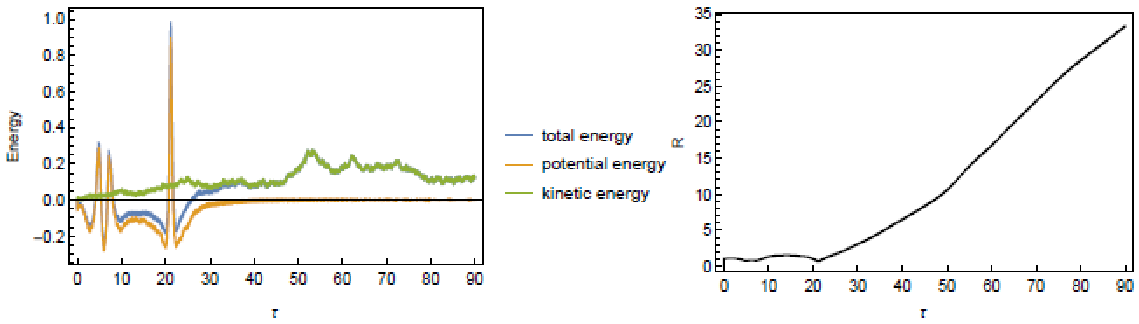
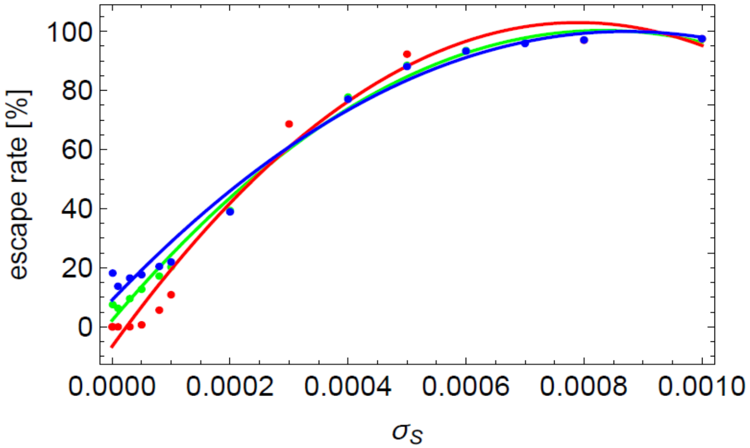


Figure 44. Brownian Motion in the Störmer Problem: energy (left) and distance (right) for a sample trajectory with $X_0 = 0.7$, $Y_0 = 0.8$, $Z_0 = 0$, $V_{x0} = 0.01$, $V_{y0} = 0.1$, $V_{z0} = 0.1$, $h = 0.001$, $L = 150000$, $\sigma_S = 10^{-3}$, $\Gamma = 10^{-3}$

Escape rate = percent of escape trajectories out of 10^4 identically set trajectories

Escape trajectory = trajectory for which the last 10^3 steps are well fitted ($R^2 > 0.9$) by a straight line

Brownian Motion in the Classical Stormer Problem – Escape rate



Brownian Motion in the Störmer Problem: escape rate as a function of noise magnitude for an ensemble of trajectories for $h = 0.001$, $L = 150000$, $\Gamma = 10^{-3}$, $X_0 = 0.7$, $Y_0 = 0.8$, $Z_0 = 0$, with $V_{x0} = 0.01$, $V_{y0} = 0.1$, $V_{z0} = 0.1$ green, $V_{x0} = 0.1$, $V_{y0} = 0$, $V_{z0} = 0$ red, and $V_{x0} = 0.1$, $V_{y0} = 0.1$, $V_{z0} = 0.1$ blue. Each ensemble has N_{traj} realizations and σ_S is the only parameter which varies between ensembles. The fitting function has the equation $-1.40 \times 10^8 \sigma_S^2 + 2.34 \times 10^5 \sigma_S + 3.09$ for the green fit, $-1.75 \times 10^8 \sigma_S^2 + 2.77 \times 10^5 \sigma_S - 6.2$ for the red fit and $-1.19 \times 10^8 \sigma_S^2 + 2.08 \times 10^5 \sigma_S + 9.07$ for the blue fit.

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Thank you!